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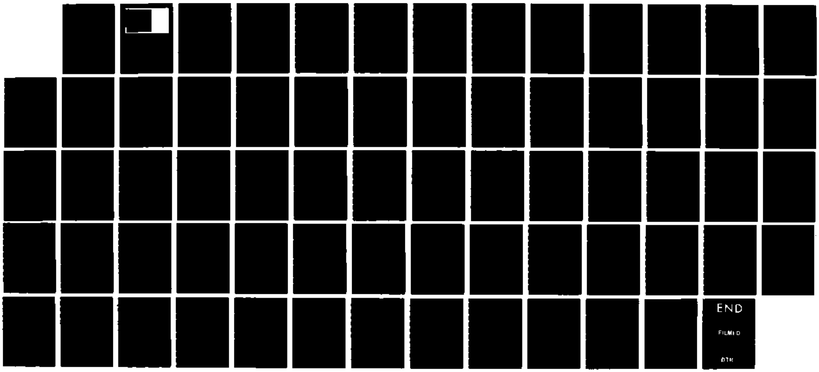

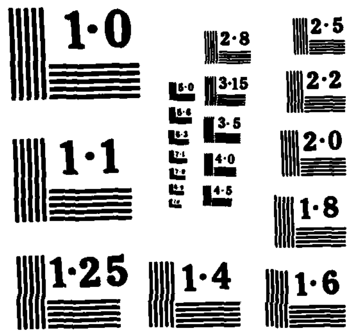

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DECAY WITH A RATE FOR NONCOMPACTLY SUPPORTED SOLUTIONS OF CONSERVATION LAWS

Blake Temple

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

This document

~~We~~ shows that solutions of the Cauchy problem for systems of two conservation laws decay in the supnorm at a rate that depends only on the L^1 (Sub 1) norm of the initial data. This implies that the dissipation due to the entropy dominates the nonlinearities in the problem at a rate depending only on the L^1 (Sub 1) norm of the initial data. Our results apply to any BV initial data u_0 satisfying $u_0(\pm\infty) = 0$, and $\mathrm{Sup}\{u_0(\cdot)\} \ll 1$. The problem of decay with a rate independent of the support of the initial data is central to the issue of continuous dependence in systems of conservation laws because of the scale invariance of the equations. Indeed, our result implies that the constant state is stable with respect to perturbations in L^1_{loc}. This is the first stability result in an L^p norm for systems of conservation laws. It is crucial that we estimate decay in the supnorm since the total variation does not decay at a rate independent of the support of the initial data.

The main estimate requires an analysis of approximate characteristics for its proof. A general framework is developed for the study of approximate characteristics, and the main estimate is obtained for an arbitrary number of equations.

AMS (MOS) Subject Classifications: 65M10, 76N99, 35L65, 35L67

Key Words: Riemann Problem, Random Choice Method, Decay, Stability, Continuous Dependence, Conservation Laws, Cauchy Problem. nonlinear partial differential equations,

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

A system of two conservation laws in one dimension is a set of first order nonlinear partial differential equations of the form

$$(1)\qquad \begin{aligned} u_t + f(u,v)_x &= 0 \ , \\ v_t + g(u,v)_x &= 0 \ , \end{aligned}$$

where (u,v) is a vector function of (x,t), $x \in R$, $t > 0$. The Cauchy problem asks for a solution of (1) given the "initial" values of u and v at time $t = 0$. Equations of type (1) arise, for example, in gas dynamics where they express the conservation of quantities like mass, momentum and energy, when diffusion is neglected. Typically, smooth solutions of (1) cannot be found. This is due to the formation of shock waves. Shock waves are the mechanism by which entropy is dissipated in solutions of (1). Moreover, this mechanism is isolated in equations of type (1) since this is the only dissipative mechanism occurring in solutions of (1). The results in this paper imply that the dissipation of entropy is a dominant effect in the sense that it forces solutions to decay to zero at an estimable rate.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

DECAY WITH A RATE FOR NONCOMPACTLY SUPPORTED SOLUTIONS OF CONSERVATION LAWS

Blake Temple

§1. INTRODUCTION

Consider the Cauchy problem for a system of n conservation laws

$$u_t + f(u)_x = 0 \quad ,$$
(1)
$$u(x,0) = u_0(x) \quad ,$$

where $u \equiv (u_1,\ldots,u_n)$, and $x \in \mathbb{R}$, $t \in \mathbb{R}^+$. We study decay and continuous dependence in solutions of (1) which are obtained as limits of approximate solutions generated by the random choice method of Glimm [6]. Thus we are interested in solutions that take values in a neighborhood U of some constant state $\tilde{u}$. We assume that df, the matrix derivative of f, is smooth, has real and distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ in U, and that $\nabla\lambda_p \cdot R_p > 0$ in U [9]. Here R_p denotes the unit right eigenvector corresponding to eigenvalue λ_p. By changing the frame or translating the flux function f if necessary, we assume without loss of generality that $\tilde{u} = 0$ and $\lambda_p > 0$, $p = 1,\ldots,n$.

Let $u(x,t)$ denote a weak solution of (1) which is a limit of approximate solutions generated by the random choice method. The main result of this paper is the following theorem which is proved in the case $n = 2$:

THEOREM (1) For every $V > 1$ and $0 < \sigma < 1$ there exists constants $\delta = \delta(V) < 1$ and $C(\sigma) > 1$ such that, if $u_0(\cdot)$ satisfies

(2) $$u_0(\pm\infty) = 0 \quad ,$$

(3) $$TV\{u_0(\cdot)\} < V \quad ,$$

and

(4) $$\|u_0(\cdot)\|_S < \delta \quad ,$$

then

(5) $$\|u(\cdot,t)\|_S < C(\sigma)\left\{\log\left[\frac{t}{\|u_0(\cdot)\|_{L^1}}\right]\right\}^{-\frac{1}{2+\sigma}}$$

for all $t > \|u_0(\cdot)\|_{L^1}$. Here constants depend only on f and their arguments, $\|\ \|_S$ denotes supnorm and $\|\ \|_{L^1}$ denotes L^1 norm; i.e.

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$$\|u_0(\cdot)\|_S \equiv \sup_{-\infty<x<+\infty} |u_0(x)| \quad ,$$

$$\|u_0(\cdot)\|_{L^1} \equiv \int_{-\infty}^{\infty} |u_0(x)|\,dx \quad .$$

If we fix the initial data $u_0(\cdot)$ and let $t \to \infty$, then (5) gives the decay of the solution $u(\cdot,t)$ in the supnorm at a rate independent of the support of the data. Said differently, (5) verifies that the dissipation in solutions of (1) due to increasing entropy overcomes the nonlinearities in the problem at a rate depending only on the L^1-norm of the initial data [cf [9]]. If we fix t and take a sequence of initial data tending to zero in L^1, then (5) gives a rate at which the supnorm at time t tends to zero with the L^1 norm of the initial data. Because the values of u at time t have a bounded domain of dependence, (5) also gives a rate at which $u(\cdot,t)$ tends to zero in $L^1_{\ell oc}$ as the initial data tends to zero in $L^1_{\ell oc}$. This is the first continuous dependence result for systems in the norm L^1.

Other decay results for systems have been obtained by Glimm/Lax, DiPerna and Liu [4, 5, 7, 10-14]. For these results decay is obtained by means of estimates for the decay of the total variation. In the case of nonperiodic initial data, a rate of decay in the total variation is obtained only in the presence of compactly supported data, and the rate depends on the support of the data. It is crucial in (5) that we estimate the decay in the supnorm instead of the total variation norm because simple examples show that the total variation does not decay at a rate that depends only on the L^1 norm of the initial data.

Our interest in the L^1 norm in (5) stems from an interest in the problem of stability, by which we mean the problem of the continuous dependence of solutions on the initial data. To put the issue of stability into perspective, we make the following definitions: we say that solutions of (1) are strongly stable in a norm $\|\ \|$ if there is a constant $C > 0$ such that

$$\|u(\cdot,t) - v(\cdot,t)\| < C\|u(\cdot,0) - v(\cdot,0)\| \quad , \tag{6}$$

for all weak solutions u and v. We say that solutions of (1) are weakly stable in $\|\ \|$ with a rate if

$$(7) \qquad \|u(\cdot,t) - v(\cdot,t)\| \le F(\|u(\cdot,0) - v(\cdot,0)\|) \,,$$

for all weak solutions u and v, where F is a fixed function satisfying $\lim_{\xi\to\infty} F(\xi) = 0$. It is well known that solutions to scalar conservation laws are L^1-contractive [cf 8]. In the language above this says that solutions of a scalar conservation law are strongly stable in L^1 with constant $C = 1$. Moreover, by studying solutions containing a single shock, it is apparent that solutions of (1) are not weakly stable in the supnorm or in the total variation norm, and are not strongly stable in any L^p, $p > 1$. This leaves L^1 as a leading candidate for studying stability in systems of conservation laws. Estimate (5) proves that the constant state is weakly stable in $L^1_{\ell oc}$. As a further comment, in [22] it is proven that solutions to systems are not L^1-contractive relative to a constant state in any metric that is compatible with the u-space topology. This directly implies that there is no metric D and constant $\omega > 0$ for which the following Gronwall type estimate holds in any neighborhood of $\bar{u}$:

$$\frac{d}{dt}\int_{-\infty}^{\infty} D(u(x,t),\bar{u})\,dx \le \omega \int_{-\infty}^{\infty} D(u(x,0),\bar{u})\,dx \;.$$

Thus, (5) gives the stability of the constant state in $L^1_{\ell oc}$ in a regime where a Gronwall inequality fails in some essential way. It is an open problem whether the constant state is strongly stable in $L^1_{\ell oc}$.

In the case of periodic initial data, Theorem (1) holds with $u_0(\cdot)$ replaced by the initial data in one period. Thus (5) gives a decay in L^∞ at a rate independent of the period. Again, for periodic initial data, the total variation does not decay at a rate depending only on the L^1 norm of the data, and for the previous decay results the rate of decay in the total variation depends on the length of the period in the initial data. Our methods also give directly that periodic data decays like $(\frac{P}{t})^{1/4}$ where P is the length of a period. This however, is not sharp in light of the t^{-1} decay rate obtained by Glimm/Lax [7].

We now indicate the *proof* of Theorem (1). Theorem (1) is obtained by estimating the decay of the quadratic functional Q which was constructed by Glimm in [6]. *Specifically*, let h denote a mesh length in x, and let $u^h(x,t)$ denote a corresponding approximate

solution generated from initial data $u_0(\cdot)$ by the random choice method. Roughly speaking, the values of u^h at time t are obtained by approximating the actual solution by a set $\{\gamma_\ell\}$ of simple waves each of which moves at close to characteristic speed. The function $Q(t)$ is defined by

$$Q(t) \equiv \Sigma|\gamma_\ell|\ |\gamma_m| \quad , \tag{8}$$

where, again roughly, the sum is over all pairs of waves at time t that will interact at some later time due to differences in the wave speeds. (In the words of Glimm, summed over all "approaching" waves.) Here, $|\gamma_\ell|$ in (8) denotes the strength of the wave γ_ℓ (for details see sections (2) and (3)). In [6] it is proved that Q is a positive decreasing function of time. Heuristically, this is because a term is lost from the sum in (8) whenever two waves cross each other in the xt-plane. The functional Q measures the potential for interaction of waves, but contains no information regarding the time at which interactions will occur. Theorem (1) is a corollary of the following technical lemma which is a sharp estimate for the rate at which Q decreases as a function of the supnorm and the L^1 norm. For this lemma, assume that $n = 2$ (or that there exists a coordinate system of Riemann invariants).

<u>LEMMA</u> (2A): Let Q denote the quadratic functional associated with an arbitrary approximate solution u^h which is generated from initial data $u_0(\cdot)$ that satisfies conditions (2) - (4) of Theorem (1). Then there exists a constant $G > 0$ depending only on f and V and a mesh length $h_0 = h_0(\varepsilon,M)$ such that, if

$$\|u^h(\cdot,0)\|_S > \frac{1}{M} \quad , \tag{9}$$

$$\|u^h(\cdot,0)\|_{L^1} = \varepsilon \quad , \tag{10}$$

and

$$h < h_0 \quad , \tag{11}$$

then

$$Q(0) - Q(\varepsilon(GM)^2) > \frac{1}{(GM)^2} \quad . \tag{12}$$

In words, (12) states that Q will decrease by an amount on the order of the supnorm squared in a time which is on the order of the L^1-norm divided by the supnorm squared. For the case $n > 2$ we obtain (12) under the assumption that the total variation of $u_0(\cdot)$ is small [cf. 6].

LEMMA (2B): If $n > 2$, then there exists $V << 1$ such that, if $u_0(\cdot)$ satisfies (2) and (3) of Theorem (1), then the conclusions of Lemma (2A) hold.

In the case of periodic initial data, Lemmas (2) hold with $u_0(\cdot)$ replaced by the initial data in each period.

The proof of Lemma (2) is given in section 6. (See Theorem 6.3 for a detailed restatement of Lemma 2.) The proof is quite technical and uses the theory of wave tracing. The theory of wave tracing was developed by Liu to prove that the random choice method converges weakly so long as the sample sequence is equidistributed [15]. Wave tracing is a method of keeping track of left and right states on approximate characteristics [1, 4, 5, 7, 10, 15, 16]. Previous decay results for systems use the theory of approximate characteristics, but rely on global mechanisms and do not require keeping track of left and right states. (*It is important, however, to recognize that in* [5], these methods are localized, and decay in Q is used to control decay in the total variation for non-compactly supported data. Of course, no rate can be obtained for decay in the total variation.) Here we develop the theory of wave tracing from what we believe is a simpler set of definitions and a simpler notation than has been previously given. The presentation is general, and essentially self contained. Motivations for the constructions can be found in [15, 16].

We now deduce Theorem (1) from Lemma (2A) using the basic results of Glimm. The remainder of this paper is then devoted to the proof of Lemma (2). We first give a precise statement of the results in [6]. (See [23] for a proof of the supnorm estimates.)

LEMMA (GL): Assume the u_0 satisfies the conditions (2), (3) and (4) of Theorem (1). Then each approximate solution u^h is defined for every $h > 0$ and $t > 0$ and, moreover, there exists $G_0 > 0$ such that

$$TV\{u^h(\cdot,t)\} \leq G_0 \; TV\{u_0(\cdot)\} \quad , \tag{13}$$

$$\|u^h(\cdot,t)\|_S \leq G_0\|u_0(\cdot)\|_S \quad , \tag{14}$$

$$\|u^h(\cdot,t_2) - u^h(\cdot,t_1)\|_{L^1} \leq G_0\{h + |t_2-t_1|\} \quad , \tag{15}$$

and

$$Q(t_2 - Q(t_1) \leq 0 \tag{16}$$

for all $t_1 < t_2$. (From here on out we use G_0 to denote a generic constant that depends only on V and f.)

In the case of arbitrary n, the results in [6] are that (13), (15) and (16) hold so long as V is sufficiently small. The reason we can obtain (5) in the case $n = 2$ and not $n > 2$ is that we use (14), a result that is not known for $n > 2$. (For $n > 2$ one can show by our methods that there exists a sequence of times $t_j \to \infty$ for which (5) holds.)

So assume that Lemma (2A) and the assumptions of Theorem (1) hold. Let $M > 1$ be given. We estimate the time at which $\|u^h(\cdot,t)\|_S \leq \frac{1}{M}$ for $h \leq h_0$. Let $G_0 > 1$ be large enough so that $Q(0) \leq G_0$. Set

$$G_1 \equiv G_0G \tag{17}$$

and let

$$N \equiv [(G_1M)^2] + 1 \tag{18}$$

where [ ] denotes "greatest integer in". Define the times $t_1,\ldots,t_N$ between which Q decreases by an amount $(G_1M)^{-2}$ as follows:

$$t_0 \equiv 0 \quad , \tag{19}$$

$$t_{n+1} \equiv \sup\{t > t_n : Q(t_n+) - Q(t) \leq \frac{1}{(G_1M)^2}\} \quad .$$

Define

$$\varepsilon_n \equiv \|u^h(\cdot,t_n+)\|_{L^1} \quad . \tag{20}$$

Let $m \leq N$ be that smallest integer for which

$$t_{m+1} - t_m > \varepsilon_m(G_1M)^2 \tag{21}$$

if such an integer exists. Otherwise let $m = N$. Now if $t_m < \infty$, then Lemma (2A) implies that

$$\|u^h(\cdot,t_m+)\|_S \leq \frac{1}{G_0M} \quad . \tag{22}$$

To see this, note that the contrapositive of Lemma (2A) states that if $Q(0) - Q(\varepsilon(GM)^2) < \frac{1}{(GM)^2}$, then $\|u^h(\cdot,0)\|_S < \frac{1}{M}$. If $t_m < \infty$, then this applies with $u^h(\cdot,0)$ replaced by $u^h(\cdot,t_m+)$ and M replaced by G_0M. (Note that if $m = N$ and $t_N < \infty$, then (21) holds in this case because Q can incur no more than N decreases of magnitude $(GM)^{-2}$.) Thus if $t_m < \infty$, then by (14)

$$\|u^h(\cdot,t)\|_S < \frac{1}{M}$$

for all $t > t_m$ as stated.

It remains only to estimate t_m. We show that

$$t_m < (G_2M)^{(G_2M)^2}\varepsilon_0 \tag{23}$$

where $G_2 = 2G_0G_1$. Without loss of generality, assume that $t_1 > \varepsilon$ and $t_1 > h$ so that (15) gives

$$\varepsilon_n < G_0t_n \quad , \tag{24}$$

where again we take G_0 to be generic. Thus if $n < m$,

$$t_{n+1} - t_n < \varepsilon_n(G_1M)^2 < G_0t_n(G_1M)^2 \quad , \tag{25}$$

so that

$$t_{n+1} < \{1 + G_0(G_1M)^2\}t_n \quad . \tag{26}$$

By induction this implies that

$$\begin{aligned} t_m &< \{1 + G_0(G_1M)^2\}^{m-1}t_1 \\ &< \{1 + G_0(G_1M)^2\}^{m-1}\varepsilon_0(G_1M)^2 \\ &< \{1 + G_0(G_1M)^2\}^m\varepsilon_0 \\ &< \{1 + G_0(G_1M)^2\}^N\varepsilon_0 \qquad (27) \\ &< \{1 + G_0(G_1M)^2\}^{(G_1M)^2}\varepsilon_0 \\ &< (G_2M)^{(G_2M)^2}\varepsilon_0 \end{aligned}$$

where $G_2 = 2G_0G_1$. Thus we have that

$$\|u^h(\cdot,t)\|_S < \frac{1}{M} \tag{28}$$

so long as

$$t > (G_2M)^{(G_2M)^2}\varepsilon_0 \ .$$

In particular, let $t = (G_2M)^{(G_2M)^2}$ and choose $0 < \sigma < 1$. Then

(29) $$\log[\frac{t}{\varepsilon_0}] = (G_2M)^2\log(G_2M) < (C(\sigma)M)^{2+\sigma}$$

for some $C(\sigma) > 0$. Thus

$$C(\sigma)\{\log(\frac{t}{\varepsilon_0})\}^{-\frac{1}{2+\sigma}} > \frac{1}{M} > \|u^h(\cdot,t)\|_S \ .$$

Now if u is a limit of approximate solutions u^h as $h \to 0$, then it follows that $\|u^h(\cdot,0)\|_{L^1} \to \|u_0(\cdot)\|_{L^1}$, and we conclude that

(31) $$\|u(\cdot,t)\|_S < C(\sigma)\{\log[\frac{t}{\|u_0(\cdot)\|_{L^1}}]\}^{-\frac{1}{2+\sigma}} \ .$$

This completes the proof of Theorem (1).

In the case of periodic data, the estimate (31) is obtained in the same manner by replacing u_0 by the initial data in one period. In this case $C(\sigma)$ is independent of the period. For periodic data we can also use Lemma (2) to obtain a rate of decay which depends on the period, since for periodic data,

(32) $$\|u^h(\cdot,t)\|_{L^1} < G_0P$$

where P is the length of one period. In this case we can use (32) instead of (24) and argue as follows: if $u^h(\cdot,t) > \frac{1}{G_0M}$, then Lemma (2) gives

$$Q(t) - Q(t + G_0P(G_1M)^2) > \frac{1}{(G_1M)^2} \ .$$

Since $Q(0) < G_0$, we see that Q can incur only $G_0(G_1M)^2$ decreases of magnitude $\frac{1}{(G_1M)^2}$ before $Q(t) < \frac{1}{(G_1M)^2}$, in which case

$$\|u^h(\cdot,t)\|_S < \frac{1}{G_0M} \ .$$

Thus $\|u_h(\cdot,t)\|_S$ must be smaller than $\frac{1}{G_0M}$ before time $T = G_0P(G_1M)^2G_0(G_1M)^2 \equiv CPM^4$.

This gives that

$$\|u^h(\cdot,t)\|_S < \frac{1}{G_0 M}$$

for all $t > CPM^4$, in which case

(33) $$\|u^h(\cdot,t)\|_S < C\left(\frac{t}{P}\right)^{-1/4} .$$

We note that (33) is not sharp (t^{-1} is sharp, cf. Glimm/Lax [7]). This might be expected since we are not invoking global mechanisms of decay as in [7].

The remainder of this paper is devoted to the proof of lemmas (2) and (3). Before embarking on the proof, we briefly discuss the idea behind it. The idea is that, since $u_0(\pm\infty) = 0$, if $\|u_0(\cdot)\|_S = \frac{1}{M}$ and $\|u_0(\cdot)\|_{L^1} = \varepsilon$, then there must be a "spike" in the initial data of height on the order of $\frac{1}{M}$ in $|u|$ and width in x on the order of εM. For example, if $|u_0(x)| > \frac{1}{2M}$ for all $x \in [x_A, x_B]$, then $\varepsilon = \|u_0(\cdot)\|_{L^1} > |x_B - x_A| \frac{1}{2M}$, which implies

$$|x_B - x_A| < 2M\varepsilon .$$

Thus consider the case

$$u_0(x) = \begin{cases} \bar{u} & x_A < x < x_B \\ 0 & \text{otherwise} \end{cases}$$

where $|\bar{u}| = \frac{1}{M}$, $x_B - x_A = \varepsilon M$. This data resolves into four simple waves associated with the Riemann problems $[0,\bar{u}]$ and $[\bar{u},0]$ [cf. §2, [9]). Label these waves α_1, α_2, β_1 and β_2 as in figure 1.

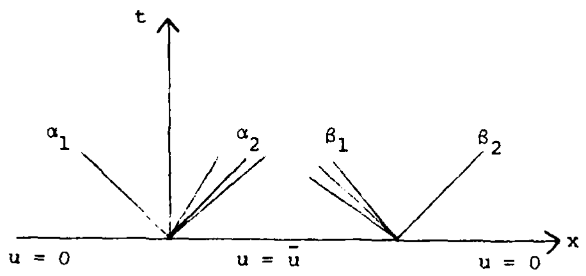

Figure 1

has been defined in such a way that properties (4.1) - (4.3) are satisfied. We define $M_p(J+1)$, $\Gamma_p(J+1)$ in terms of $M_p(J)$, $\Gamma_p(J)$ and see that properties (4.1) - (4.3) hold for $M_p(J+1)$, $\Gamma_p(J+1)$, respectively.

To define $M_p(J+1)$ and $\Gamma_p(J+1)$, we define the sets

$$M_p^i(J+1) = \{\ell \in M_p(J+1) : \ell(J) = i\} \quad ,$$

and

$$\{\gamma_\ell\}_{\ell \in M_p^i(J+1)} \quad ,$$

for each $i \in \mathbb{Z}$. The sets $M_p(J+1)$ and $\Gamma_p(J+1)$ are then defined by

$$M_p(J+1) \equiv \{M_p(J) \cap \tilde{M}(J-1)\} \cup \{\bigcup_i M_p^i(J+1)\} \quad ,$$

$$\Gamma_p(J+1) \equiv \{\gamma_\ell : \ell \in M_p(J+1)\} \quad .$$

So fix $i \in \mathbb{Z}$. Define

(4.12) $$L_* \equiv \{\ell \in M_p(J) : \ell(J) = i,\ \mathrm{sgn}(\ell) \neq \mathrm{sgn}(\gamma_{ij}^p)\} \quad .$$

If $\gamma_{iJ}^p = 0$, then define

$$M_p^i(J+1) \equiv L_* \quad ,$$

$$\Gamma_p^i(J+1) \equiv \{\gamma_\ell : \ell \in L_*\} \quad .$$

So assume $\gamma_{iJ}^p \neq 0$. Let $\{\ell_0,\ldots,\ell_{a-1}\} \subseteq M_p(J)$ denote the indices in $M_p(J)$ satisfying

$$\ell_k(J) = i \quad ,$$

$$\mathrm{sgn}(\ell_k) = \mathrm{sgn}\{\gamma_{ij}^p\} \quad ,$$

and ordered so that $\ell_{k-1} < \ell_k$, $1 < k < a-1$, in the sense of Property (4.1). Define

$$u_L \equiv L(\gamma_{iJ}^p) \quad ,$$

$$u_R \equiv R(\gamma_{ij}^p) \quad ,$$

$$\sigma_0 = 0,\ \sigma_k = \sum_{s=0}^{k-1} \gamma_{\ell_s}(t_J-),\ k = 0,\ldots,a \quad ,$$

$$u_k \equiv T(\sigma_k; u_L) \quad .$$

Property (4.3) expresses the fact that characteristics trace nonzero elementary waves of a given family and sign:

PROPERTY (4.3): If $\ell \in M_p(J)$, then the signed strength $\gamma_\ell(t)$ of the characteristic γ_ℓ is constant and nonzero in $[t_\ell^0, t_\ell^1)$. We write

(4.11) $$\operatorname{sgn}\{\gamma_\ell(t)\} \equiv \operatorname{sgn}(\ell) .$$

For convenience, we set

$$\gamma_\ell(t) = 0$$

for $t \notin [t_\ell^0, t_\ell^1)$.

We now define $M_p(J)$ and $\Gamma_p(J)$ by induction on J. We simultaneously verify properties (4.1) - (4.3) which are assumed in the induction step.

First assume $J = 1$. Define

$$M_p(1) = \{\ell_i^q : \gamma_{i0}^{pq} \neq 0,\ p = 1,2,\ q = L,R\} ,$$

where ℓ_i^q is defined by

$$\ell_i^q(0) = i ,$$

$$\ell_i^q(1) = \begin{cases} i & \text{if } q = L \\ i+1 & \text{if } q = R . \end{cases}$$

Then for $0 \leq t < k \equiv t_1$, and $\gamma_{i0}^{pq} \neq 0$, define

$$u^L_{\ell_i^q}(t) = L\left(\gamma_{i0}^{pq}\right) ,$$

$$u^R_{\ell_i^q}(t) = R\left(\gamma_{i0}^{pq}\right) .$$

It is easy to verify that properties 1, 2 and 3 hold for $M_p(1)$ and $\Gamma_p(1)$.

Now assume for induction that $M_p(J) \subseteq \tilde{M}(J)$ has been defined, and that for every $\ell \in M_p(J)$,

$$\gamma_\ell[t] = (u_\ell^L(t), u_\ell^R(t))$$

$\sigma_R \equiv \gamma_{ij}^p$ (so that, e.g. $u_R = T_p(\sigma_R; u_L)$). Then for $t_j \le t < t_{j+1}$,

$$u^L_{\ell_0}(t) = u_L \quad ,$$

$$u^R_{\ell_a}(t) = u_R \quad ,$$

and

$$u^R_{\ell_{k-1}}(t) = u^L_{\ell_k}(t) = T(\sigma_k; u_L) \quad ,$$

where $\sigma_k \in (0,\sigma_R)$ and $|\sigma_{k+1}| > |\sigma_k|$, $1 \le k \le a-1$. We define

$$\gamma_{\ell_k}(t) \equiv \sigma_{k+1} - \sigma_k \quad , \tag{4.6}$$

(i.e., round brackets around t to distinguish it from $\gamma_\ell[t]$) to be the signed strength of the characteristic γ_{ℓ_k} at time $t \in [t_j, t_{j+1})$, $1 \le k \le a-1$. Moreover, for $\sigma \in (0,1)$ and $t \in [t_j, t_{j+1})$, we define

$$u^\sigma_{\ell_k}(t) \equiv T(\sigma_k + \sigma(\sigma_{k+1} - \sigma_k);\ u_L) \tag{4.7}$$

for $1 \le k \le a-1$. Note that if Property (4.2) holds, then (4.6) and (4.7) define $\gamma_\ell(t)$ and $u^\sigma_\ell(t)$ for all $\ell \in M_p(J)$, $t < t_J$. We use (4.7) to define two characteristics $\gamma_{\ell^L_\sigma}$ and $\gamma_{\ell^R_\sigma}$ corresponding to each $\sigma \in (0,1)$ and each $\ell \in M_p(J)$ satisfying $\ell(J) = i$, as follows:

$$\ell^q_\sigma(j) = \begin{cases} \ell(j) & \text{if } j \le J-1 \\ i & \text{if } j = J,\ q = L \\ i+1 & \text{if } j = J,\ q = R \quad , \end{cases} \tag{4.8}$$

$$\gamma_{\ell^L_\sigma}[t] = (u^L_\ell(t), u^\sigma_\ell(t)) \quad , \tag{4.9}$$

$$\gamma_{\ell^R_\sigma}[t] = (u^\sigma_\ell(t), u^R_\ell(t)) \quad . \tag{4.10}$$

(I.e., σ determines a splitting of the characteristic γ_ℓ into a characteristic $\gamma_{\ell^L_\sigma}$ of strength $\sigma\,\gamma_\ell(t)$ and a characteristic $\gamma_{\ell^R_\sigma}$ of strength $(1-\sigma)\gamma_\ell(t)$).

$$M(J) = \bigcup_{p=1}^{n} M_p(J) \quad ,$$

the RHS being a disjoint union. Corresponding to each $\ell \in M_p(J)$ is the characteristic

$$\gamma_\ell = (u_\ell^L, u_\ell^R) \in \Gamma_p(J) \quad ,$$

each entry being a function of t for $t_\ell^0 < t < t_\ell^1 < t_J$. We let $\gamma_\ell[t] \equiv (u_\ell^L[t], u_\ell^R[t])$ denote the value of γ_ℓ at time t. The functions $u_\ell^L[t]$, $u_\ell^R[t]$ (the left and right states of the characterisitcs γ_ℓ at time t) are constant on intervals $t_j < t < t_{j+1}$ for $j_\ell^0 < j < j_\ell^1$. $\Gamma(J)$ is the disjoint union

$$\Gamma(J) = \bigcup_{p=1}^{n} \Gamma_p(J) \quad .$$

For convenience we set $\gamma_\ell[t] = 0$ for $t \notin [t_\ell^0, t_\ell^1)$.

Before defining $M(J)$ and $\Gamma(J)$ precisely, we first list three properties (properties (4.1), (4.2) and (4.3)) which the characteristics satisfy. Then for $p = 1,\ldots,n$ we simultaneously define $M_p(J)$, $\Gamma_p(J)$, and verify properties (4.1) - (4.3) by induction on J.

Property (4.1) states that each set $M_p(J)$, $p = 1,\ldots,n$ is partially ordered, and expresses the fact that characteristics of the same family never cross:

<u>PROPERTY</u> (4.1): If ℓ_1, $\ell_1 \in M_p(J)$, then for every

$$j \in [j_{\ell_1}^0, j_{\ell_1}^1] \cap [j_{\ell_2}^0, j_{\ell_2}^1] \quad , \tag{4.4}$$

we have that either $\ell_1(j) < \ell_2(j)$ or $\ell_2(j) < \ell_1(j)$. If $[j_{\ell_1}^0, j_{\ell_2}^1] \cap [j_{\ell_2}^0, j_{\ell_1}^1] \neq \emptyset$ and $\ell_1(j) < \ell_2(j)$ for j satisfying (4.4), then we say that $\ell_1 < \ell_2$ (or ℓ_1 lies to the left of ℓ_2).

Property (4.2) states that the left and right states of the p-characteristics $\ell \in M_p(J)$ that satisfy $\ell(j) = i$, $j < J$, partition the p-wave curve $T_p(\cdot, L(\gamma_{ij}^p))$ between $L(\gamma_{ij}^p)$ and $R(\gamma_{ij}^p)$:

<u>PROPERTY</u> (4.2): Let $i \in Z$, $j \in [0,J-1]$ be such that $\gamma_{ij}^p \neq 0$. Let $\{\ell_0,\ldots,\ell_{a-1}\}$ be the set of p-characteristics $\ell \in M_p(J)$ satisfying $\ell(j) = i$, and ordered so that $\ell_k < \ell_{k+1}$ in the sense of Property (4.1). Let $u_L \equiv L(\gamma_{ij}^p)$, $u_R \equiv R(\gamma_{ij}^p)$ and let

characteristics in detail.

Let

$$M(j^0,j^1) = \{j^0,j^0+1,\dots,j^1\}$$

where $j^0 < j^1$ are in the set $\{0,1,\dots,J\}$. Let

(4.1) $$\tilde{M}(j^0,j^1) = \{\ell : M(j^0,j^1) \to \mathbb{Z} : \ell(j) - \ell(j-1) \in \{0,1\}\} \quad .$$

For each element of $\ell \in \tilde{M}(j^0,j^1)$, define the function

(4.2) $$x_\ell : [t_{j^0},t_{j^1}] \to R$$

as follows:

$$x_\ell(t_j) = x_{\ell(j)} \quad , \quad j^0 \leq j \leq j^1 \quad ,$$

$$x_\ell(t) = x_\ell(t_j) + [x_\ell(t_{j+1}) - x_\ell(t_j)]h, \; t_j \leq t \leq t_{j+1} \quad .$$

Thus each element $\ell \in \tilde{M}(j^0,j^1)$ corresponds to a continuous, timelike, piecewise linear curve in the xt-plane given by the graph of x_ℓ. Note that the graph of x_ℓ is defined in $[t_{j^0},t_{j^1}]$ and connects successive mesh points; i.e., the slope of the curve is either 0 or $\frac{k}{h}$. If j, t is outside the domain of ℓ, x_ℓ, then we write $\ell(j) = \emptyset$, $x_\ell(t) = \emptyset$, respectively.

Let

(4.3) $$\tilde{M}(J) = \bigcup_{0 \leq j^0 < j^1 \leq J} \tilde{M}(j^0,j^1) \quad .$$

Then for $\ell \in \tilde{M}(J)$, define j_ℓ^0, j_ℓ^1 to be the positive integers such that

$$\ell \in \tilde{M}(j_\ell^0,j_\ell^1) \quad .$$

Let $t_\ell^0 \equiv t_{j_\ell^0}$, $t_\ell^1 \equiv t_{j_\ell^1}$.

We presently define the set $M(J)$ of indices for the characteristics, as well as the set $\Gamma(J)$ of characteristics, by induction on J. For $p = 1,\dots,n$, the set of indices for the p-characteristics is a set

$$M_p(J) \subset \tilde{M}(J) \quad ,$$

and

§4. DEFINITION OF APPROXIMATE CHARACTERISTICS

In this section we define the set of approximate characteristics (heretofor referred to as "characteristics") associated with a given approximate solution u^h and time level T_J. In the next section, we study properties of the characteristics. The procedure is as follows. We first define an index set $\tilde{M}(J)$ for the timelike piecewise linear curves that connect successive mesh points (x_i,t_j) in the xt-plane [cf. 16]. A subset $M(J) \subseteq \tilde{M}(J)$ corresponds to the set of characteristics. We call this the index set for the characteristics. Each element $\ell \in M(J)$ gives the position of an "elementary wave" γ_ℓ at different time levels. The piecewise linear curves that are undefined at $t = t_J$ correspond to elementary waves cancelled, and those undefined at $t = 0$ correspond to elementary waves created by nonlinearities. We let $N \subseteq M(J)$ denote the index set for all such characteristics [cf. 15].

We define the elementary wave γ_ℓ associated with $\ell \in M(J)$ by assigning a left state $u_\ell^L[t]$ and right state $u_\ell^R[t]$ to each time level that intersects the piecewise linear curve defined by ℓ. We define $\gamma_\ell[t] \equiv (u_\ell^L[t], u_\ell^R[t])$, we call $\Gamma(J) = \bigcup_{\ell \in M(J)} \{\gamma_\ell\}$ the set characteristics defined for time t_J. The assignment of states to characteristic curves is done as follows: We first state three properties that the assignment should satisfy, and then we assume the properties to hold in order to define the characteristics at the induction step. Thus the characteristics are defined and the properties are verified simultaneously at the induction step.

The characteristics determine a partitioning of the waves in u^h appearing before time t_J. It is important to estimate the "fineness" of the partition. For Liu [15], the fineness is built into the procedure by an initial partitioning of the waves. The cost of having $M(J)$ as an index set is that we must estimate the fineness of the partition as a function of J. This together with an estimate for the speed of characteristic curves is given in Theorem (5.12). The remainder of section 5 is essentially devoted to obtaining estimates for the change $|u_\ell^q[t] - u_\ell^q[0]|$, $q = L,R$, in terms of changes in Q and in terms of strengths of elementary waves assigned to characteristic curves that cross the characteristic curve of ℓ in time $[0,t_J]$. We now proceed with the definition of the

then u^h is well defined, takes values in U for all time, and moreover

(3.18) $$V_j \le G_0 V_0 \quad ,$$

(3.19) $$G_0^{-1} \sum_i D_{ij} \le Q(t_j-) - Q(t_j+) \le G_0 \sum_i D_{ij}$$

and

(3.20) $$\|u^h(\cdot,t) - u^h(\cdot,s)\|_{L^1} \le G_0\{h + |t-s|\}$$

for all $j, s, t > 0$.

__LEMMA__ (GL3). Assume that $u_0(\pm\infty) = 0$ and that there exists a coordinate system of Riemann invariants (eg $n = 2$). Then for every $V > 0$ there exists $\tilde{M} > 0$ such that if

(3.21) $$V_0 \le V \quad ,$$

and

(3.22) $$\|u_0^h\|_S \le \frac{1}{\tilde{M}} \quad ,$$

then u^h is well defined for all $t \ge 0$, takes values in U, and (3.18) - (3.20) hold together with

(3.23) $$\|u^h(\cdot,t)\|_S \le G_0\|u^h(\cdot,0)\|_S \quad .$$

(See [23] for a proof of Lemma (GL3).)

Two waves γ^p_{ij} and $\gamma^{p'}_{i'j}$ are said to approach at time t_j if one of two conditions holds [6]:

$$p < p' \text{ and } i > i' , \tag{3.10A}$$

or

$$p = p' \text{ and either } \gamma^p_{ij} < 0 \text{ or } \gamma^{p'}_{i'j} < 0 . \tag{3.10B}$$

For $t_j < t < t_{j+1}$, define [4]

$$Q(t) \equiv \Sigma |\gamma^p_{ij}| \, |\gamma^{p'}_{i'j}| \tag{3.11}$$

where the sum is over all pairs of waves γ^p_{ij} and $\gamma^{p'}_{i'j}$ that approach at time t_j.

Define

$$D_{ij} \equiv \Sigma |\gamma^{pR}_{i-1,j-1}| \, |\gamma^{p'L}_{i,j-1}| \tag{3.12}$$

where the sum is over all pairs of approaching waves that enter the diamond Δ_{ij}, and finally define the cancellation

$$C^p_{ij} \equiv \frac{1}{2} \{ |\gamma^{pR}_{i-1,j-1}| + |\gamma^{pL}_{i,j-1}| - |\gamma^{pR}_{i-1,j-1} + \gamma^{pL}_{i,j-1}| . \tag{3.13}$$

Then C^p_{ij} measures the amount of p-wave cancelled from both $\gamma^{pR}_{i-1,j-1}$ and $\gamma^{pL}_{i,j-1}$ at the interaction in Δ_{ij}.

The following lemmas are due to Glimm [4]. Let

$$V_j \equiv \sum_{i,p} |\gamma^p_{ij}| \tag{3.14}$$

estimate the total variation of $u^h(\cdot,t)$ for $t_j < t < t_{j+1}$.

<u>LEMMA</u> (GL1). If u^h U_0, then for all i, j,

$$|\gamma^p_{ij} - \gamma^{pR}_{i-1,j-1} - \gamma^{pL}_{i,j-1}| < G_0 D_{ij} . \tag{3.14}$$

This immediately implies

$$|\gamma^p_{ij}| = |\gamma^{pR}_{i-1,j-1}| + |\gamma^{pL}_{i,j-1}| - 2C^p_{ij} + 0(1)D_{ij} \tag{3.16}$$

with $|0(1)| < G_0$. (Again, G_0 denotes a generic constant.)

<u>LEMMA</u> (GL2). *There exists a constant* $V > 0$ *and a number* $G_0 > 0$ *such that if*

$$V_0 < V , \tag{3.17}$$

$$L(\gamma_{ij}^{pL}) = L(\gamma_{ij}^{p})$$

$$R(\gamma_{ij}^{pL}) = u^h(x_i + a_{j+1}h,\ t_{j+1}-) = L(\gamma_{ij}^{pR}) \tag{3.7}$$

$$R(\gamma_{ij}^{pR}) = R(\gamma_{ij}^{p}) \quad .$$

More generally, if (3.6) does not hold, then define

$$\gamma_{ij}^{pL} \equiv \begin{cases} \gamma_{ij}^{p} \quad \text{if} \quad a_j > \lambda_p(R(\gamma_{ij}^{p})) & , \ \gamma_{ij}^{p} > 0 \\ \gamma_{ij}^{p} \quad \text{if} \quad a_j > s(\gamma_{ij}^{p}) & , \ \gamma_{ij}^{p} < 0 \\ 0 \quad \text{otherwise} \end{cases} \tag{3.8}$$

$$\gamma_{ij}^{pR} \equiv \begin{cases} \gamma_{ij}^{p} \quad \text{if} \quad a_j < \lambda_p(L(\gamma_{ij}^{p})) & , \ \gamma_{ij}^{p} > 0 \\ \gamma_{ij}^{p} \quad \text{if} \quad a_j < s(\gamma_{ij}^{p}) & , \ \gamma_{ij}^{p} < 0 \\ 0 \quad \text{otherwise} \end{cases} \tag{3.9}$$

[Here, for example, γ_{ij}^{pL} *is defined to be zero* if the wave lies to the right of the sample point $x_i + a_jh$, and γ_{ij}^{p} if it lies to the left].

By construction, the waves γ_{ij}^{p} solve the Riemann problem for $u_L = u^h(x_{i-1} + a_{j-1}h,\ t_j) \equiv L(\gamma_{ij}^{1})$, $u_R = u^h(x_i + a_{j-1}h,\ t_j) \equiv R(\gamma_{ij}^{n})$ that is posed at (x_i, t_j) in the approximate solution $u^h(x,t)$. Because we assume that all wave speeds are positive, the waves γ_{ij}^{p} are formed by the interaction of the waves $\gamma_{i-1,j-1}^{pR}$ with the waves $\gamma_{i,j-1}^{pL}$ at time t_{j-1}, $p = 1,2$. To emphasize this, we let Δ_{ij} denote the diamond of interactions centered at (x_i, t_j) [6, 16]; i.e., consisting of vertex points $(x_{i-1} + a_{j-1}h,\ t_j)$, $(x_i + a_{j-1}h,\ t_j)$, $(x_i, t_j - \frac{1}{2}k)$ and $(x_i, t_j + \frac{1}{2}k)$. We say that the waves γ_{ij}^{p} which cross the upper wedge of the diamond are formed due to the interaction of the waves $\gamma_{i-1,j-1}^{pR}$ and $\gamma_{i,j-1}^{pL}$ which cross the lower wedge of the diamond. We call γ_{ij}^{p} a wave that leaves the diamond Δ_{ij}, and we call the nonzero waves among $\gamma_{i-1,j-1}^{pR}$ and $\gamma_{i,j-1}^{pL}$ the waves that enter the diamond Δ_{ij}.

§3. THE RANDOM CHOICE METHOD APPROXIMATES

In this section we define the approximate solutions of (1) generated by the random choice method of Glimm [6], and develop notation required for the subsequent sections.

Let h be a mesh length in x, and let

$$k = Ch \tag{3.1}$$

be the corresponding mesh length in t, $C > \sup_{u \in U_1} \{|\lambda_n(u)|\}$. For $i, j \in \mathbb{Z}$, $j > 0$, let $x_i \equiv ih$, $t_j \equiv jk$. Let $\underset{\sim}{a}$ be a sample sequence, $\underset{\sim}{a} \equiv \{a_j\}_{j=1}^{\infty}$, $0 < a_j < 1$. For given initial data $u_0(\cdot) \subset U$, define the random choice method approximate solution $u^h(x,t) \equiv u^h(x,t;\underset{\sim}{a})$ by induction on j as follows: First, for $x_i < x < x_{i+1}$, define

$$u^h(x,0) \equiv u_0^h(x) = u_0(x_i + \frac{h}{2}) \quad . \tag{3.2}$$

Next, assume for induction that $u^h(x,t)$ has been defined for $t < t_j$. Define

$$u^h(x,t_j) \equiv u^h(x_i + a_j h, t_j -) \quad , \tag{3.3}$$

and for $t_j < t < t_{j+1}$, define $u^h(x,t)$ to be the solution of the Riemann problem posed in (3.3) at time t_j. By (3.1), u^h is well defined so long as $u^h(x,t_j) \subseteq U$ for all t_j.

Let u^h be any approximate solution that is well defined by the above procedure. Let γ_{ij}^p denote the name as well as the signed strength of the p-wave that appears in the solution of the Riemann problem that is posed at (x_i,t_j) in the approximate solution u^h. Define

$$L(\gamma_{ij}^p) \equiv \text{left state of the wave } \gamma_{ij}^p \quad ,$$

$$R(\gamma_{ij}^p) \equiv \text{right state of the wave } \gamma_{ij}^p \quad .$$

If γ_{ij}^p is a shock wave, define

$$s(\gamma_{ij}^p) \equiv \text{speed of wave } \gamma_{ij}^p \quad . \tag{3.5}$$

If γ_{ij}^p is a rarefaction wave, then the wave is "split" at time t_{j+1} if

$$\lambda_p(L(\gamma_{ij}^p)) \frac{k}{h} < a_j < \lambda_p(R(\gamma_{ij}^p)) \frac{k}{h} \quad . \tag{3.6}$$

In this case define γ_{ij}^{pL} and γ_{ij}^{pR} by

by the condition

$$u_{k+1} \in \mathcal{V}_k(u_k), \quad 1 \leq k \leq n \quad .$$

We call $\mathcal{T}_p$ [cf [11] a wave strength measure for the p-simple waves if, for each $u_L \in \mathcal{U}_1$, $\mathcal{T}_p(\sigma;u_L)$ is a parameterization of $\mathcal{V}_p(u_L)$ in $\mathcal{U}_1$, $\sigma \in R$, and moreover $\mathcal{T}_p \in C^2$ with bounded third derivatives with respect to both arguments, $\mathcal{T}_p(0;u_L) = u_L$ and

(2.3) $$\frac{\partial \lambda_p(\mathcal{T}_p(\sigma,u_L))}{\partial \sigma} > 0 \quad .$$

If $L(\gamma^p) = u_L$ and $R(\gamma^p) = \mathcal{T}_p(\sigma;u_L)$, then we define the signed strength of γ^p to be σ. We let $\gamma^p \equiv \sigma$ so that "γ^p" denotes both the name as well as the signed strength of the wave. By (2.3), $\gamma^p < 0$ for shocks and $\gamma^p > 0$ for rarefaction waves.

In the following three sections we study the random choice method approximates and associated approximate characteristics using arbitrary $\mathcal{T}$ to define wave strength. In section 6 we define $\mathcal{T}$ by means of a best approximation to a coordinate system of Riemann invariants. The following lemma is a direct consequence of the fact that the speed s of a shock is equal to the average characteristic speed to within terms that are quadratic in the strength of the shock [9, 11, 17]:

<u>**LEMMA**</u> (2.1). Assume that $u_R = \mathcal{T}_p(\sigma;u_L)$, $\sigma < 0$. Then the speed s of the corresponding shock wave satisfies

$$s = \lambda_p\left(\frac{u_L + u_R}{2}\right) + O(1)\sigma^2 \quad .$$

§2. THE RIEMANN PROBLEM

The Riemann problem is the initial value problem (1) where the initial data has the form

$$u_0(x) = \begin{cases} u_L & \text{for} \quad x < 0 \\ u_R & \text{for} \quad x > 0 \quad . \end{cases}$$

We assume that the Riemann problem is uniquely solvable by the method of Lax [9] for all u_L and u_R in a neighborhood U of $u = 0$. In particular, assume that all states that appear in the solution of a Riemann problem posed in U lie in a set $U_1 \supseteq U$. Assume that $\lambda_p(u) < \lambda_{p'}(v)$ for all $u, v \in U_1$, $1 < p < p' \leq n$, and that (1) is genuinely nonlinear in all characteristic fields throughout U_1.

Let $R_p(u_L)$ denote the integral curve of R_p that contains the point u_L. Let $R_p^+(u_L)$ denote the p-rarefaction curve associated with the point u_L; i.e., that portion of $R_p(u_L)$ for which $\lambda_p(u) > \lambda_p(u_L)$. By [9] there exists a unique curve $S_p(u_L)$ that makes C^2 piecewise C^3 contact with $R_p(u_L)$ at the point u_L, and such that for each $u \in S_p(u_L)$ there is a scalar s such that

$$s[u] = [f] \quad , \tag{2.1}$$

where $[u] \equiv u - u_L$, $[f] = f(u) - f(u_L)$. Statement (2.1) is the Rankine Hugoniot jump condition, and we say that $S_p(u_L)$ is in the Hugoniot locus of u_L. Let $S_p^-(u_L)$ denote the p-shock curve associated with u_L; i.e., that portion of $S_p^-(u_L)$ for which $\lambda_p(u) < \lambda_p(u_L)$. We assume that λ_p is monotone on both $S_p(u_L)$ and $R_p(u_L)$, so that the curve

$$\mathcal{Y}_p(u_L) \equiv S_p^-(u_L) \cup R_p^+(u_L) \tag{2.2}$$

is a C^2 piecewise C^3 curve throughout U_1. For $u_R \in \mathcal{Y}_p(u_L)$, the Riemann problem is solved by a p-simple wave: a shock wave if $\lambda_p(u_R) < \lambda_p(u_L)$ and a rarefaction wave if $\lambda_p(u_R) > \lambda_p(u_L)$ [9, 17]. We let γ^p denote any p-simple wave. For given γ^p, let $L(\gamma^p)$ denote the left state of γ^p and let $R(\gamma^p)$ denote the right state. For arbitrary states $u_L, u_R \in U$, the Riemann problem is solved uniquely by n simple waves $\gamma^1,\ldots,\gamma^n$ which are separated in the xt-plane (going from left to right) by the constant states $u_1 \equiv u_L,\ldots,u_{n+1} \equiv u_R$. The states u_i, and hence the waves γ^p, are uniquely determined

Finally in section 6 we give the proof of lemmas 2 and 3.

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It takes the longest time for waves to interact if $\alpha_2 = 0$ or $\beta_1 = 0$. Assume that $\alpha_2 = 0$ and to be specific, assume that α_1 is a shock wave. In this case, β_1 must be a rarefaction wave because $u = 0$ is both the left most and right most state in the problem. (Here we use the assumption that the waves are weak.) Now we can estimate the time at which Q must decrease by order $\frac{1}{M^2}$; i.e., the time it takes α_1 to interact with β_1. But the differences in speeds between α_1 and β_1 at time zero is on the order of $\frac{1}{M}$, so interaction must occur within a time on the order of T where T satisfies

$$|x_B - x_A| - \frac{1}{M} T = 0 \quad ;$$

i.e.,

$$T = M|x_B - x_A| = M^2\varepsilon \quad .$$

Since two waves α_1 and β_1 of strength on the order of $\frac{1}{M}$ interact before time T, by (8) we expect Q to decrease by an amount on the order $\frac{1}{M^2}$ in time T. This is

$$Q(0) - Q(T) = 0(1)\frac{1}{M^2} \quad ,$$

which is (12).

We implement the above idea as follows: given data $u_0(\cdot)$, we locate p-waves α and β whose speeds at time $t = 0$ differ by $0(1)\frac{1}{M}$ and whose x distance apart at $t = 0$ is $0(1)\ \varepsilon M$ as above. We identify these waves at a later time by means of approximate characteristics. We then assume for contradiction that Q does not decrease by $0(1)\frac{1}{M^2}$ in time $0(1)\ \varepsilon M^2$. A consequence of this is that α and β may be chosen so that the corresponding characteristics do not intersect before time $0(1)\ \varepsilon M^2$. We finally derive a contradiction by estimating that since the decrease in Q is small, the speed of the characteristics α and β agree with the speeds at time zero sufficiently to guarantee that they intersect before time $0(1)\ \varepsilon M^2$. By this contradiction we can conclude the proof of lemmas (2) and (3). In the above analysis we must keep track of left and right states on approximate characteristics. This is essentially the wave tracing idea of Liu [15].

In sections 2 and 3 we review the Riemann problem and the random choice method and establish notation. In section 4 we define approximate characteristics. In section 5 we establish properties of approximate characteristics. (This is done in a general setting.)

Let σ_R be defined by

$$u_R \equiv T(\sigma_R; u_L) \quad .$$

Note that σ_k is defined for $0 \leq k \leq a$, and

$$\gamma_{\ell_k}(t_J-) = \sigma_{k+1} - \sigma_k, \; k = 0,\ldots,a-1 \quad .$$

Choose n so that

$$\sigma_R \in (\sigma_n, \sigma_{n+1}] \quad .$$

If $\sigma_R > \sigma_a$, let $\sigma_{a+1} \equiv \sigma_R$, so that then $n = a$. Let

$$u_M = \begin{cases} u_h(x_i + a_{J+1}h, t_{J+1}) & \text{if } \gamma_{iJ}^{PL} \neq 0 \text{ and } \gamma_{iJ}^{PR} \neq 0 \\ u_L & \text{if } \gamma_{iJ}^{PL} = 0 \\ u_R & \text{if } \gamma_{iJ}^{PR} = 0 \end{cases} ,$$

and let σ_M be define by

$$u_M = T(\sigma_M; u_L) \quad .$$

Now for $0 \leq k < n$, define

$$\ell_k'(j) \equiv \begin{cases} \ell_k(j) & \text{if } j \leq J \\ i & \text{if } j = J+1, \; \sigma_k < \sigma_M \\ i+1 & \text{if } j = J+1, \; \sigma_k \geq \sigma_M \end{cases} ,$$

$$\gamma_{\ell_k'}[t] \equiv \begin{cases} \gamma_{\ell_k}[t] & \text{if } t < t_J \\ (u_k, u_{k+1}) & \text{if } t_J \leq t < t_{J+1} \end{cases} .$$

For $k = n$, define

$$\ell_n^L(j) \equiv \begin{cases} 0 & \text{if } j \leq J, \; n = a \\ \ell_n(j) & \text{if } j \leq J, \; n < a \\ i & \text{if } j = J+1, \; \sigma_n < \sigma_M \\ i+1 & \text{if } j = J+1, \; \sigma_n \geq \sigma_M \end{cases} ,$$

$$\ell_n^R(j) \equiv \begin{cases} 0 & \text{if } j < J , \\ i & \text{if } j = J , \\ i & \text{if } j = J+1,\ \sigma_n < \sigma_M , \\ i+1 & \text{if } j = J+1,\ \sigma_n > \sigma_M , \end{cases}$$

$$\gamma_{\ell_n^L}[t] \equiv \begin{cases} 0 & \text{if } t < t_J ,\ n = a , \\ \gamma_{\ell_n}[t] & \text{if } t < t_J ,\ n \leq a-1 , \\ (u_n,u_R) & \text{if } t_J \leq t < t_{J+1} , \end{cases}$$

$$\gamma_{\ell_n^R}[t] \equiv \begin{cases} 0 & \text{if } t_j < t_J \\ (u_R,u_{n+1}) & \text{if } t_J \leq t < t_{J+1} . \end{cases}$$

Let

$$L_{<n} \equiv \{\ell_k' : 0 \leq k < n\} ,$$

$$L_n \equiv \{\ell_n^L, \ell_n^R\} ,$$

$$L_{>n} \equiv \{\ell_k : n < k \leq a-1\} .$$

Define

(4.13) $$L^* \equiv \{\ell \in L_{<n} \cup L_n : \gamma_\ell(t_J+) \neq 0\} .$$

For $\ell \in L^*$ let σ_ℓ^q, $q = L, R,$ be defined by

$$u_\ell^q(t_J+) = T(\sigma_\ell^q; u_L) ,$$

and set

(4.14) $$L_{\neq M} \equiv \{\ell \in L^* : \sigma_M \notin (\sigma_\ell^L, \sigma_\ell^R)\} .$$

Moreover if $\ell \in L^*$ and $\sigma_M \in (\sigma_\ell^L, \sigma_\ell^R)$, then let

$$\sigma \equiv \frac{\sigma_M - \sigma_\ell^L}{\sigma_\ell^R - \sigma_\ell^L} ,$$

and set (cf. (4.8) - (4.10))

(4.15) $$L_M \equiv \{\ell^q_\sigma : \ell \in L^*,\ \sigma_M \in (\sigma^L_\ell, \sigma^R_\ell),\ q = L \text{ or } R\} \ .$$

Finally, we define

(4.16) $$M^i_p(J+1) \equiv L_{\neq M} \cup L_M \cup L_{>n} \cup L_* \ ,$$

in the case $\gamma^p_{iJ} \neq 0$.

Since γ_ℓ is defined for all $\ell \in M^i_p(J+1)$ this completes the definition of $M^i_p(J+1)$ and $\Gamma^i_p(J+1)$. We leave it to the reader to verify from the above construction that Properties (4.1) - (4.3) are satisfied by $M_p(J+1)$ and $\Gamma_p(J+1)$. This completes the definition of the approximate characteristics.

We say that $\gamma_\ell \in \Gamma(J)$ is cancelled at (x_i, t_j) if $\ell(j) = i$ and $j = j^1_\ell$. From (4.16) it is clear that $\gamma_\ell \in \Gamma(J+1)$ is cancelled at (x_i, t_J) if and only if $\ell \in L_{>n} \cup L_*$ or $\ell = \ell^R_n$ where $n = q$.

§5. PROPERTIES OF APPROXIMATE CHARACTERISTICS

In this section we study properties of the characteristics $\Gamma(J)$ and index set $M(J)$ defined in the previous section. Let u^h be a given approximate solution generated by the random choice method from initial data u_0. The sets $\Gamma(J)$ and $M(J)$ associated with u^h are determined by the choices of u_0, J, a, h and T, which we take to be given. We let $M \equiv M(J)$, $\Gamma \equiv \Gamma(J)$. Throughout the remainder of the paper, $\sum_L$ is taken to mean $\sum_{\ell \in L}$ whenever $L \subseteq M$. We now develop some definitions.

Let $N(t)$ denote the index set for the "null" characteristics that are either cancelled or else are created by "nonlinearities" in time $[0,t]$, $t < t_J$; i.e.,

$$N(t) \equiv \{\ell \in M : t_\ell^0 > 0 \ \text{ or } \ t_\ell^1 < t\} \quad . \tag{5.1}$$

We partition $N(t)$ into $N_0(t)$ and $N_\emptyset(t)$ as follows:

$$N_0(t) \equiv \{\ell \in N(t) : t_\ell^0 = 0\} \quad , \tag{5.2}$$

$$N_\emptyset(t) \equiv \{\ell \in N(t) : t_\ell^0 > 0\} \quad . \tag{5.3}$$

By Property (4.3), the index set M partitions into

$$M = M^+ \cup M^-$$

where

$$M^+ \equiv \{\ell \in M : \operatorname{sgn}(\ell) > 0\} \quad ,$$

$$M^- \equiv \{\ell \in M : \operatorname{sgn}(\ell) < 0\} \quad .$$

We call M^+ [resp M^-] the set of rarefaction wave [resp. shock wave] characteristics. For $p = 1,\ldots,n$ and $q = +$ or $-$, define

$$M_p^q \equiv M_p \cap M_q \quad . \tag{5.4}$$

Similarly, define

$$N_p^q(t) \equiv N(t) \cap M_p^q \quad ,$$

and set

$$N_{p0}^q(t) \equiv N_p^q(t) \cap N_0(t) \quad ,$$

$$N_{p\emptyset}^q(t) \equiv N_p^q(t) \cap N_\emptyset(t) \quad ,$$

so that $N_p^q(t)$ is the disjoint union of $N_{p0}^q(t) \cup N_{p\emptyset}^q(t)$.

We say that two characteristics γ_ℓ and γ_m intersect at time t_j if $\ell(j) = m(j)$ and $\ell(j-1) \neq m(j-1)$. The following lemma which can be verified by induction on J implies the uniqueness of intersection times for characteristics in different families.

LEMMA (5.1). If $\ell \in M_p$, $m \in M_{p'}$, and $p < p'$, then $x_m - x_\ell$ is a nondecreasing function of time for $t \in [t_\ell^0, t_\ell^1] \cap [t_m^0, t_m^1]$.

The next lemma follows from Properties (4.1) through (4.3):

LEMMA (5.2). If $\ell, m \in M_p$, $\mathrm{sgn}(\ell) \neq \mathrm{sgn}(m)$, and ℓ intersects m at time $t_j < t < t_J$, then either $\ell \in N(t)$ or $m \in N(t)$. Moreover, if ℓ and m are both shock wave characteristics, then there is at most one intersection time t_0, and for all $t \in [t_0, t_\ell^1) \cap [t_0, t_m^1)$ we have

$$x_\ell(t) = x_m(t) \quad .$$

Define

$$N_{ij}(\ell) \equiv \begin{cases} |\gamma_\ell(t_j+)| - |\gamma_\ell(t_j-)| & \text{if } \ell(j) = i \quad , \\ 0 & \text{otherwise} \quad . \end{cases} \tag{5.6}$$

We call $N_{ij}(\ell)$ the nonlinearity contributed by the characteristic γ_ℓ at the mesh point (x_i, t_j) [cf. 16]. In particular, since $\gamma_\ell(t)$ is constant on $[t_\ell^0, t_\ell^1]$, we must have $N_{ij}(\ell) = 0$ unless $\ell \in N(t_j)$ and either $j_\ell^0 = j$ or $j_\ell^1 = j$. Define

$$C_{ij}(\ell) = \begin{cases} |\gamma_\ell(t_j-)| & \text{if } \ell(j) = i \text{ and } j = j_\ell^1 \\ 0 & \text{otherwise} \quad , \end{cases} \tag{5.7}$$

$$E_{ij}(\ell) = \begin{cases} |\gamma_\ell(t_j+)| & \text{if } \ell(j) = i \text{ and } j = j_\ell^0 \\ 0 & \text{otherwise} \quad , \end{cases} \tag{5.8}$$

so that

$$N_{ij}(\ell) = -C_{ij}(\ell) + E_{ij}(\ell) \quad . \tag{5.9}$$

We call $C_{ij}(\ell)$ the cancellation and $E_{ij}(\ell)$ the error in the nonlinearity $N_{ij}(\ell)$. The following proposition is a restatement of Lemma (GL1) in the language of approximate characteristics:

PROPOSITION (5.3). There exists $G_0 > 0$ such that

(5.10) $$\sum_{M} E_{ij}(\ell) < G_0 D_{ij} \quad ,$$

and

(5.11) $$\left| \{ \sum_{M_p^q} C_{ij}(\ell) \} - C_{ij}^p \right| < G_0 D_{ij}$$

for $p = 1,\ldots,n$ and $q = +$ or $-$.

(Note that since $E_{ij}(\ell) = C_{ij}(\ell) = 0$ for $\ell \notin N(t_j)$, sums over M and M_p^q can be replaced by sums over $N(t_j)$ and $N_p^q(t_j)$ in (5.10) and (5.11), respectively.)

COROLLARY (5.3). There exists $G_0 > 0$ such that

(5.12A) $$\sum_{N_\emptyset(t)} |\gamma_\ell| < G_0[Q(0) - Q(t)] \quad ,$$

(5.12B) $$\sum_{M} |\gamma_\ell| < G_0 V_0 \quad .$$

Proof. If $\ell \in N_\emptyset(t)$, then $t_\ell^0 > 0$. Thus by (5.5),

$$|\gamma_\ell| = E_{\ell(j_\ell^0),j_\ell^0}(\ell)$$

and so by Lemma (GL1)

$$\sum_{N_\emptyset(t)} |\gamma_\ell| < \sum_{ij} \sum_{M} E_{ij}(\ell) < \sum_{ij} G_0 D_{ij} < G_0[Q(0) - Q(t)] \quad .$$

This verifies (5.12A). For (5.12B), note that since $|\gamma_\ell(t)|$ is constant on $[t_\ell^0, t_\ell^1)$, we must have

$$\sum_{M \setminus N_\emptyset(t_J)} |\gamma_\ell| < G_0 V_0 \quad .$$

Thus by (5.12A)

$$\sum_{M} |\gamma_\ell| = \sum_{M \setminus N_\emptyset(t_J)} |\gamma_\ell| + \sum_{N_\emptyset(t_J)} |\gamma_\ell| < G_0 V_0 \quad .$$

For $j < J$, define the set of indice pairs associated with waves that approach at time t [cf. (3.11)]:

$$A(t) = A^1(t) \cup A^2(t)$$

where

(5.13)
$$A^1(t) \equiv \{\langle \ell,m\rangle : \ell,\ m \in M_p \ \text{ for some } \ p = 1,\dots,n,\ x_\ell(t) < x_m(t) \ , \ \text{and either } \ell \text{ or } m \text{ is in } M^-\} \ ,$$

(5.14) $$A^2(t) \equiv \{\langle \ell,m\rangle : \ell \in M_p \ , m \in M_{p'}, \ \text{ for } \ p < p', \ \text{ and } \ x_\ell(t) > x_m(t)\} \ .$$

(Here we use $\langle \ell,m\rangle$ to denote the set or unordered pair $\{\ell,m\}$. Note also that $x_\ell(t) < x_m(t)$ implies $x_\ell(t) \neq \emptyset \neq x_m(t)$, so that $|\gamma_\ell(t)| \neq 0 \neq |\gamma_m(t)|$).

We call $A^1(t)$ and $A^2(t)$ the index sets for the characteristics that approach at time t and are in the same and different families, respectively. Define

(5.15) $$A_{ij} \equiv \{\langle \ell,m\rangle : \ell(j) = m(j) = i \ \text{ and } \ \langle \ell,m\rangle \in A(t_j-)\} \ .$$

The following is a consequence of Property (4.3) together with the definition of Q and D_{ij}:

<u>LEMMA</u> (5.4): We have

(5.16) $$Q(t) = \sum_{\langle \ell,m\rangle \in A(t)} |\gamma_\ell| \ |\gamma_m| \ ,$$

and

(5.17) $$D_{ij} = \sum_{\langle \ell,m\rangle \in A_{ij}} |\gamma_\ell| \ |\gamma_m| \ .$$

From here on out, we write $\sum\limits_A$ in place of $\sum\limits_{\langle \ell,m\rangle \in A}$, etc.

Define $A_0(t)$ and $A_\emptyset(t)$ as follows:

(5.18) $$A_0(t) \equiv \{\langle \ell,m\rangle \in A(0) : \text{either } \ \ell \ \text{ or } \ m \ \text{ is in } \ N_0(t)\} \ ,$$

(5.19) $$A_\emptyset(t) \equiv \{\langle \ell,m\rangle \in A(t) : \text{either } \ \ell \ \text{ or } \ m \ \text{ is in } \ N_\emptyset(t)\} \ .$$

<u>LEMMA</u> (5.5): Let $A' \subseteq A(0) \setminus A(T)$, $T < t_J$. Then

(5.20) $$A(T) \subseteq A(0) \cup A_\emptyset(t) \setminus A' \ .$$

<u>Proof</u>: By the definition of $A(t)$, $\langle \ell,m\rangle \in A(t)$ implies that $\gamma_\ell(t) \neq 0$ and $\gamma_m(t) \neq 0$, thus $A(T) \cap A' = \emptyset$. Assume then that ℓ and m are in $M \setminus N_\emptyset(T)$ and $\langle \ell,m\rangle \in A(T)$. It remains only to show that $\langle \ell,m\rangle \in A(0)$. But by the definition of A, the only way $\langle \ell,m\rangle$ could fail to be in $A(0)$ is if ℓ and m intersect at two distinct times in $[0,T]$, which contradicts Lemma (5.1) or (5.2). For example, if $\ell \in M_p$ and $m \in M_{p'}$, $p < p'$,

then by (5.14), $x_\ell(T) > x_m(T)$. Thus by Lemma (5.1), $x_\ell(t) > x_m(t)$ for all $t < T$. Hence $\langle \ell, m\rangle \in A(0)$.

<u>LEMMA</u> (5.6). There exists $G_0 > 0$ such that, for all $T < t_J$,

(5.21) $$\sum_{A_\emptyset(T)} |\gamma_\ell|\ |\gamma_m| < G_0[Q(0) - Q(T)] \quad .$$

<u>Proof</u>: By (5.12) and (5.13),

$$\sum_{A_\emptyset(T)} |\gamma_\ell|\ |\gamma_m| < \sum_{\ell \in M} |\gamma_\ell| \sum_{m \in N_\emptyset} |\gamma_m| < G_0[Q(t) - Q(0)] \quad .$$

<u>LEMMA</u> (5.7). There exists $G_0 > 0$ such that, if

(5.22) $$A' \subseteq A(0) \setminus A(T) \quad , \quad T < t_J \quad ,$$

and

(5.23) $$\sum_{A'} |\gamma_\ell|\ |\gamma_m| > L \quad ,$$

then

(5.24) $$Q(0) - Q(T) > \frac{1}{G_0} L \quad .$$

<u>Proof</u>: By Lemma (5.5) and (5.22) we have

$$A(T) \subseteq A(0) \cup A_\emptyset(T) \setminus A' \quad .$$

Thus

$$Q(T) = \sum_{A(T)} |\gamma_\ell|\ |\gamma_m| < \sum_{A(0)} |\gamma_\ell|\ |\gamma_m| + \sum_{A_\emptyset(T)} |\gamma_\ell|\ |\gamma_m| - \sum_{A'} |\gamma_\ell|\ |\gamma_m|$$

$$< Q(0) + G_0[Q(0) - Q(T)] - L \quad .$$

Combining terms, this gives

$$Q(0) - Q(T) > \frac{1}{G_0 + 1} L \quad .$$

Since G_0 is generic, this verifies (5.24).

<u>PROPOSITION</u> (5.8). There exists $G_0 > 0$ such that, if

(5.25) $$\sum_{N(T)} |\gamma_\ell| > L$$

for $T < t_J$, then

(5.26) $$Q(0) - Q(T) > \frac{1}{G_0} L^2 \quad .$$

Since T is fixed here, we set $N_p^q \equiv N_p^q(T)$, etc.

Proof: Assume that (5.25) holds. Then for some $p \in \{1,\ldots,n\}$ and $q = +$ or $-$ we must have

(5.27) $$\sum_{N_p^q(t)} |\gamma_\ell| > \frac{L}{2n} \quad .$$

For convenience, assume $q = +$. By (5.7) and (5.8),

(5.28) $$\begin{aligned} \sum_{N_p^+(t)} |\gamma_\ell| &< \sum_{ij} \sum_{N_p^+(t)} E_{ij}(\ell) + \sum_{ij} \sum_{N_p^+(t)} C_{ij}(\ell) \\ &< \sum_{ij} G_0 D_{ij} + \sum_{ij} \sum_{N_p^+(t)} C_{ij}(\ell) \end{aligned}$$

where the sum on i, j is over $\{(i,j) : -\infty < i < +\infty,\ t_j < t\}$. But $\sum_{ij} D_{ij} <$ $G_0[Q(0) - Q(t)]$, together with (5.27) and (5.29), implies that for some $G_0 > 0$,

(5.29) $$\sum_{ij} \sum_{N_p^+(t)} C_{ij}(\ell) > \frac{L}{2n} - G_0[Q(0) - Q(t)]$$

and so by (5.11)

(5.30) $$\sum_{ij} C_{ij}^p > \frac{L}{2n} - G_0[Q(0) - Q(t)] \quad .$$

Thus we can apply (5.11) again with $q = -$ to obtain

$$\sum_{ij} \sum_{N_p^-(t)} C_{ij}(\ell) - \sum_{ij} C_{ij}^p < G_0[Q(0) - Q(t)] \quad ,$$

or

(5.31) $$\sum_{ij} \sum_{N_p^+(t)} C_{ij}(\ell) > \frac{L}{2n} - G_0[Q(0) - Q(t)] \quad .$$

Therefore, by (5.7) and (5.8),

$$\begin{aligned} \sum_{s \in N_p^-(t)} |\gamma_s| &> \sum_{ij} \sum_{s \in N_p^-(t)} C_{ij}(s) - \sum_{ij} \sum_{s \in N_p^-(t)} E_{ij}(s) \\ &> \frac{L}{2n} - G_0[Q(0) - Q(t)] \quad . \end{aligned}$$

We now have that for some $G_0 > 0$, both

(5.32A) $$\sum_{r \in N_p^+} |\gamma_r| > \frac{L}{2n} - G_0[Q(0) - Q(t)]$$

and

(5.33B) $$\sum_{s \in N_p^+} |\gamma_s| > \frac{L}{2n} - G_0[Q(0) - Q(t)] \quad .$$

By Corollary (5.3), this implies that for some $G_0 > 0$, $q = +$ and $-$, we have

(5.34) $$\sum_{N_{p0}^q} |\gamma_\ell| > \frac{L}{2n} - G_0[Q(0) - Q(t)] \quad .$$

Let $A' \equiv \{\langle r,s\rangle : r \in N_{p0}^+,\ s \in N_{p0}^-\}$. Then by (5.2), (5.5) and the definition of A, $A' \subseteq A(0)\setminus A(T)$. Moreover,

(35) $$\begin{aligned}\sum_{A'} |\gamma_\ell|\,|\gamma_m| &> \sum_{r \in N_{p0}^+} \sum_{s \in N_{p0}^-} |\gamma_r|\,|\gamma_s| > \{\frac{L}{2n} - G_0[Q(0) - Q(t)]\}^2 \\ &> \frac{L^2}{4n^2} - G_0[Q(0) - Q(t)]\end{aligned}$$

where we use the fact that $L < V_0 < G_0$ and $[Q(0) - Q(t)] < G_0$ for some $G_0 > 0$. Thus by Lemma (5.7),

$$Q(0) - Q(T) > \frac{1}{G_0 4n^2} L^2 - [Q(0) - Q(T)] \quad ,$$

or

(5.35) $$Q(0) - Q(T) > \frac{1}{8G_0 n^2} L^2 \quad .$$

Since G_0 is generic we can take $\frac{1}{8G_0 n^2}$ to be G_0, so (5.35) establishes Proposition (5.8).

<u>Corollary</u> (5.8). There exists G_0 positive so that if $T < t_J$ and

(5.36) $$Q(0) - Q(T) < L \quad ,$$

then

(5.37) $$\sum_{N(T)} |\gamma_\ell| < G_0 L^{1/2} \quad .$$

We next define index sets for characteristics that intersect a given characteristic in a given time interval. Let

$$\mathcal{B}_\ell(j) \equiv \{m \in \mathcal{M} : m(j) = \ell(j) \text{ and } \langle \ell,m\rangle \in \mathcal{A}^2(t_j-)\} \quad , \tag{5.38}$$

$$B_\ell(j) \equiv \sum_{\mathcal{B}_\ell(j)} |\gamma_m| \quad , \tag{5.39}$$

and for $t_1 \in [t_{j1-1},t_{j1})$ and $t_2 \in [t_{j2},t_{j2+1})$, let

$$B_\ell[t_1,t_2] \equiv \sum_{j1}^{j2} B_\ell(j) \quad . \tag{5.40}$$

In addition, for $\ell \in \mathcal{M}_p^-$, define

$$\mathcal{B}_{\ell R}^-(j) \equiv \{m \in \mathcal{M}_p^- : m(j) = \ell(j),\ m(j-1) > \ell(j-1)\}$$

$$\mathcal{B}_{\ell L}^-(j) \equiv \{m \in \mathcal{M}_p^- : m(j) = \ell(j),\ m(j-1) > \ell(j-1)\} \quad ,$$

$$\mathcal{B}_\ell^-(j) = \mathcal{B}_{\ell R}^-(j) \cup \mathcal{B}_{\ell L}^-(j) \quad .$$

For q = L, R or absent, define $\mathcal{B}_{\ell q}^-[t_1,t_2]$ as in (5.40). (Note here that script $\mathcal{B}$ denotes a set, while upper case B denotes a real number.)

In the next section, we use the following technical result:

<u>PROPOSITION</u> (5.9). Let

$$\mathcal{M}_p^-[i_L,i_R] \equiv \{\ell \in \mathcal{M}_p^- : \ell(0) \in [i_L,i_R]\} \quad . \tag{5.41}$$

Assume that $s \in \mathcal{M}_p^- \setminus \mathcal{N}(T)$, $s(0) = i_L$, that

$$\sum_{\mathcal{M}_p^-[i_L,i_R]} |\gamma_\ell| > 4L \tag{5.42}$$

and that

$$|\gamma_{i0}^p| < L \tag{5.43}$$

for all $i \in [i_L,i_R]$ such that $\gamma_{i0}^p < 0$. Then there exists $G_0 > 0$ such that, if $L < \mu$ and

$$\sum_{\mathcal{B}_{pR}^-[0,T]} |\gamma_\ell| > L \quad , \tag{5.44}$$

then

$$Q(0) - Q(T) > G_0^{-1}L^2 \quad . \tag{5.45}$$

Moreover, if the hypotheses are satisfied with $s(0) = i_R$ and $\mathcal{B}^-_{pL}[0,T] > L$, then again (5.45) follows.

To prove Proposition (5.9), we use the following lemma:

LEMMA (5.9). Let $\{L_i\}_{i=1}^m$ be a nonnegative sequence of numbers which satisfies

(5.46)
$$0 \le L_i \le L \quad ,$$

(5.47)
$$\sum_{i=1}^{m} L_i \ge 3L \quad .$$

Then

(5.48)
$$\sum_{1\le i<j\le m} L_i L_j \ge L^2 \quad .$$

Proof. Let $i^* = \inf\{i : \sum_{i=1}^{i^*} L_i \ge L\}$. Then $L_{i^*} \le L$ implies that

$$\sum_{i=i^*+1} L_i \ge M \quad .$$

Therefore,

$$\sum_{1\le i<j\le m} L_i L_j \ge \left(\sum_{i=1}^{i^*} L_i\right)\left(\sum_{i=i^*+1}^{m} L_i\right) \ge L^2 \quad .$$

Proof of Proposition (5.9): Let

$$\mathcal{B}_0 \equiv \mathcal{B}^-_{pR}[0,T] \setminus N(T)$$

$$\mathcal{B}_\emptyset \equiv \mathcal{B}^-_{pR}[0,T] \cap N(T) \quad .$$

Let

$$L_0 = \sum_{\mathcal{B}_0} |\gamma_\ell| \quad ,$$

$$L_\emptyset = \sum_{\mathcal{B}_\emptyset} |\gamma_\ell| \quad .$$

Without loss of generality, assume $L_0 \ge 3L$. If not, then $L_\emptyset \ge L$, in which case Proposition (5.8) implies that

$$Q(0) - Q(T) \ge \frac{1}{G_0} L^2 \quad ,$$

which gives (5.45). Let

$$\mathcal{B}_{i0} \equiv \{\ell \in \mathcal{B}_0 : \ell(0) = i\} \quad ,$$

and define

$$L_i = \sum_{\mathcal{B}_{i0}} |\gamma_\ell| \quad .$$

Let $i'' = \max\{i : L_i \neq 0\}$, and let $i' = \min\{i'', i_R\}$. By (5.42) and (5.44),

$$\sum_{i=i_L}^{i''} L_i \geq 3L \quad .$$

Moreover by Property (4.3), $L_i = -\gamma^P_{i0}$ if $\gamma^P_{i0} < 0$ and $L_i = 0$ otherwise, so that $L_i \leq L$ for all $i \in [i_L, i'']$. Thus the conditions of Lemma (5.9) hold, and we have

$$\sum_{i_L \leq i < j \leq i'} L_i L_j \geq L^2 \quad .$$

But by definition, $\langle \ell, m \rangle \in A'(0) \setminus A'(T)$ if $\ell, m \in \mathcal{B}_0$ and $\ell(0) \neq m(0)$. Thus

$$A(0) \setminus A(T) \geq \sum_{A'} |\gamma_\ell| \; |\gamma_m| \geq \sum_{i_L \leq i < j \leq i'} L_i L_j \geq L^2 \quad .$$

Therefore, by Lemma (5.7) we conclude

$$Q(0) - Q(T) \geq \frac{1}{G_0} L^2 \quad .$$

Finally, define $\lambda_\ell[t]$, the speed of the characteristic γ_ℓ at time t as follows: Let $\ell \in M_p$, $t \in [t_j, t_{j+1}) \cap [t^0_\ell, t^1_\ell)$, and assume that $\ell(j) = i$. Then

$$\lambda_\ell[t] \equiv \begin{cases} \lambda(\gamma^P_{ij}) & \text{if} \quad \ell < 0 \quad , \\ \lambda_p(u^L_\ell[t]) & \text{if} \quad \ell > 0 \quad . \end{cases}$$

We now estimate the change in u^L_ℓ, u^R_ℓ and λ_ℓ for $\ell \in M(J) \setminus N(J)$.

LEMMA (5.10). Let $\ell \in M(J) \setminus N(J)$. Then

(5.50) $$|u^q_\ell[t_j+] - u^q_\ell[t_j-]| \leq G_0\{B_\ell(j) + D_{\ell(j),j}\}$$

(5.51) $$|\lambda_\ell[t_j+] - \lambda_\ell[t_j-]| \leq G_0\{B_\ell(j) + B^-_\ell(j) + C_{\ell(j),j} + D_{\ell(j),j}\}$$

for $q = L, R$.

COROLLARY (5.10). Let $\ell \in M \setminus N(T)$, $t_1 < t_2 < T < t_J$. Then for $q = L$ and R,

$$|u_\ell^q[t_2] - u_\ell^q[t_1]| < G_0\{B_\ell[t_1,t_2] + [Q(t_1) - Q(t_2)]\} \quad , \tag{5.52}$$

$$|\lambda_\ell[t_2] - \lambda_\ell[t_1]| < G_0\{B_\ell[t_1,t_2] + \bar{B}_\ell[t_1,t_2] + [Q(t_1) - Q(t_2)]^{1/2}\} \quad . \tag{5.53}$$

Moreover,

$$|u_\ell^q[t_2] - u_\ell^q[t_1] < G_0V_0 \tag{5.54}$$

$$|\lambda_\ell[t_2] - \lambda_\ell[t_1]| < G_0V_0 \quad . \tag{5.55}$$

Let $\underset{\sim}{a}$ be a fixed equidistributed sequence, $0 < a_j < 1$ [15, 16]. Let $I = (c,d) C [0,1]$ and let $0 < N_1 < N_2$. Define

$$N(I,N_1,N_2) \equiv \mathrm{Card}\{j \in [N_1,N_2] : a_j \in I\} \quad .$$

The following is a result regarding equidistributed sequences [15, 16]:

LEMMA (5.11). For every $M > 1$ there exists $N_0 > 1$ such that, if $N > N_0$, then

$$\left| |I| - \frac{N(I,j,j+N)}{N} \right| < \frac{1}{M} \tag{5.56}$$

for all $j < MN$, $I = (c,d) C [0,1]$, $|I| = |d-c|$. Moreover, for the best equidistributed sequences,

$$N(M,\underset{\sim}{a}) < G_0 M^{-1} \quad , \tag{5.57}$$

where $N(M,\underset{\sim}{a})$ denotes the infemum of all such N_0 for a given $\underset{\sim}{a}$.

For $\ell \in M$, $t \in [t_\ell^0, t_\ell^1)$, define

$$E_\ell(t) = x_\ell(t) - x_\ell(t_\ell^0) - \int_{t_\ell^0}^{t} \lambda_\ell[t]dt \quad . \tag{5.58}$$

THEOREM (5.12). For each equidistributed sequence $\underset{\sim}{a}$ there exists a positive function $\delta(\mu) < 1$, $\delta(\mu) \to 0$ as $\mu \to 0$, such that the following two statements hold:

If

$$h < \delta(\mu)\nu, \; t \in [t_\ell^0, t_\ell^1), \tag{5.59}$$

then

$$|\gamma_\ell| < \mu$$

for all $\ell \in M^+(J)$ such that $t_\ell^0 = 0$, $t_\ell^1 > \nu$.

Since $z_p(u_\ell^R[0]) - z_p(u_\ell^L[0])$ is positive for $\ell \in M^+$ and negative for $\ell \in M^-$; and since z_p increases from m to $\frac{1-\delta}{M}$ going from x_A to $\bar{x}$ at time $t = 0$, we can write

(6.59) $$\bigcup_L I_\ell = \{\bigcup_R I_\ell\} \cup \{\bigcup_{L\setminus M_p} I_\ell\} \quad .$$

Since $R \subseteq R_0 \cup \{L \cap N(T)\}$, (6.57), (6.58) and (6.59) imply

(6.60) $$\bigcup_{R_0} I_\ell \supset [0, \tfrac{1}{M}] \setminus X$$

where X is a set of small measure,

(6.61) $$m\{X\} < \frac{4\delta}{M} < \frac{1}{16M} \quad ,$$

where we have used (6.22) to estimate δ.

By (5.59), if $r \in R_0$, then

(6.62) $$|\gamma_r| < \frac{1}{16M}$$

because $h < G_0^{-1}(\frac{1}{16M})^2 T$ by (6.14). Thus (6.60), (6.61) and (6.62) immediately imply the existence of $r \in R_0$ such that (6.46) holds.

Similarly, we conclude as in (6.60) that

$$\bigcup_{S_0} I_\ell \supseteq [0, \tfrac{1}{M}] \setminus X \quad ,$$

where X satisfies (6.61), which directly implies the existence of $s \in S_0$ satisfying either (6.47), (6.48) or (6.49), (6.50).

Define [cf. (5.38) - (5.40)]

(6.63) $$R_0^L \equiv \{\ell \in R_0 : x_\ell(0) < x_r(0)\} \quad ,$$

(6.64) $$S_0^R \equiv \{\ell \in S_0 : x_\ell(0) > x_s(0)\} \quad ,$$

(6.65) $$B_0 \equiv \{\ell \in M_{p'} : p' \neq p \text{ and } \gamma_\ell \text{ intersects } \gamma_s \text{ in } [0, T/2]\} \quad .$$

(6.66) $$B_0 = \sum_{B_0} |\gamma_\ell|$$

(6.67) $$\mathcal{B}_r \equiv \mathcal{B}_r[0, T/2],\ B_r \equiv \sum_{\mathcal{B}_r} |\gamma_\ell|$$

(6.68) $$\mathcal{B}_s \equiv \mathcal{B}_s[0, T/2],\ B_s \equiv \sum_{\mathcal{B}_r} |\gamma_\ell|$$

(6.69) $$\mathcal{B}_{sq}^- \equiv \mathcal{B}_{sq}^-[0, T/2],\ B_{sq}^- \equiv \sum_{\mathcal{B}_{sq}^-} |\gamma_\ell|,\ q = L, R \text{ or absent.}$$

nd either

(6.47) $$z_p(u_s^R[0]) < \frac{3}{8M}$$

and

(6.48) $$|\gamma^p_{i_s 0}| > \frac{\delta}{M} \quad ,$$

or else

(6.49) $$z_p(u_s^R[0]) \in (\frac{1}{8M}, \frac{1}{4M})$$

and there exists i_L, i_R with $i_A < i_L < i_s < i_R < i_B$ such that

(6.50) $$|\gamma^p_{i0}| < \frac{\delta}{M}$$

for all $i \in M_p^-[i_L,i_R]$, and [cf. (5.41)]

(6.51) $$\sum_{M_p^-[i_L,i_s]} |\gamma_i| > \frac{4\delta}{M} \quad ,$$

(6.52) $$\sum_{M_p^-[i_s,i_R]} |\gamma_i| > \frac{4\delta}{M} \quad .$$

Proof. We first verify the existence of $r \in R_0$ satisfying (6.46). Let

(6.53) $$L \equiv \{\ell \in M : x_\ell(0) \in [x_A,\bar{x}]\} \quad ,$$

and let I_ℓ denote the interval $[z_p(u_\ell^L[0]), z_p(u_\ell^R[0])]$. Let $|I_\ell|$ denote the length of the interval I_ℓ. Let

(6.54) $$m \equiv \mathrm{Inf}\{z(u_0^h(x)) : x \in [x_A,\bar{x}]\} \quad .$$

By (6.39) and (6.41),

(6.55) $$-M < m < \frac{\delta}{M} \quad .$$

Property (4.3) states that the approximate characteristics partition the waves in the approximate solution u^h at each time step. This together with (6.40) implies

(6.56) $$\bigcup_L I_\ell \supseteq [m, \frac{1-\delta}{M}] \quad .$$

By Lemma (6.4),

(6.57) $$\sum_{L\setminus M_p} |I_\ell| < \frac{\delta}{M} \quad ,$$

and since we assume $Q(0) - Q(T) < \frac{1}{(GM)^2}$, Corollary (5.8) implies

(6.58) $$\sum_{L\cap N(T)} |I_\ell| < \frac{\delta}{M} \quad .$$

the case when $\frac{1}{M} = -\operatorname{Inf}_x \{z_p(u_0^h(x))\}$ being handled analogously.

By (6.39) there exists a point $\bar{x} = (\bar{i} + \frac{1}{2})h$ such that

(6.40) $$z_p(u_0^h(\bar{x})) > \frac{1-\delta}{M} \quad .$$

Define i_A and i_B by

(6.41) $$\begin{aligned} i_A &\equiv \operatorname{Sup}\{i : i < \bar{i} + \tfrac{1}{2} \text{ and } z_p(u_0^h(x_i-)) < \tfrac{\delta}{M}\} \\ i_B &\equiv \operatorname{Inf}\{i : i > \bar{i} + \tfrac{1}{2} \text{ and } z_p(u_0^h(x_i+)) < \tfrac{\delta}{M}\} \quad . \end{aligned}$$

Let

(6.42) $$\begin{aligned} x_A &\equiv (i_A - \tfrac{1}{2})h \quad , \\ x_B &= (i_B + \tfrac{1}{2})h \quad . \end{aligned}$$

Define $R \subseteq M_p^-$ and $S \subseteq M_p^+$ by

(6.43) $$\begin{aligned} R &\equiv \{\ell \in M_p^+ : x_\ell(0) \in (x_1, \bar{x})\} \quad , \\ S &\equiv \{\ell \in M_p^- : x_\ell(0) \in (\bar{x}, x_2)\} \quad , \end{aligned}$$

and let [cf. (5.1)]

(6.44) $$\begin{aligned} R_0 &\equiv R \setminus N(T) \quad , \\ S_0 &\equiv S \setminus N(T) \quad . \end{aligned}$$

<u>LEMMA (6.5)</u>. The following estimate holds:

(6.45) $$|x_B - x_A| < \frac{\varepsilon M}{\delta} \quad .$$

<u>Proof</u>. By (6.40),

$$\varepsilon = \int_{-\infty}^{\infty} |u_0^h(x)|\,dx > \frac{\delta}{M}\,|x_B - x_A| \quad .$$

Solving for $|x_B - x_A|$ gives (6.45).

<u>PROPOSITION (6.6)</u>. If $Q(0) - Q(T) < \frac{1}{(GM)^2}$, then there exists $r \in R_0$ and $s \in S_0$, $s(0) = i_s$, such that the following conditions hold:

(6.46) $$\{z_p(u_r^L[0]),\ z_p(u_r^R[0])\} \subseteq (\frac{3}{4M}, \frac{7}{8M}) \quad ,$$

We use this to construct two characteristics $\gamma_r \in \Gamma_p^+$ and $\gamma_s \in \Gamma_p^-$ which would intersect before time T if there were no interactions to deflect the speeds of these characteristics from their initial speeds at time $t = 0$. We then assume that $Q(0) - Q(T) < \frac{1}{(GM)^2}$. By this assumption, there exists such γ_r, γ_s which are not cancelled in time $[0,T]$. We then use the same assumption to obtain estimates for the change in the speeds of γ_r and γ_s between time $t = 0$ and $t = T$. These estimates are sufficient to guarantee that in time $[0,T]$ the wave speeds of γ_r and γ_s are not deflected enough from their initial speeds to prevent them from intersecting before time T. This intersection implies that one of them is cancelled before time T [cf Property (4.1)]. This is a contradiction and thus we conclude that $Q(0) - Q(T) \geq \frac{1}{(GM)^2}$. We first use Lemma (6.1) together with (6.30) or (6.24) to estimate the total variation in z_p contributed by characteristics not in the p-family.

LEMMA (6.4). Our assumptions on u^h imply

$$\sum_{M\setminus M_p} |z_p(u_\ell^R[0]) - z_p(u_\ell^L[0])| \leq \frac{\delta}{M} \quad . \tag{6.36}$$

Proof: A restatement of Lemma (6.1) in the language of approximate characteristics is that

$$|z_p(u_\ell^R[0]) - z_p(u_\ell^L[0])| \leq \frac{G_0}{M} |\gamma_\ell(0)| \tag{6.37A}$$

for all $\ell \in M\setminus M_p$; and if z is a coordinate system of Riemann invariants, then

$$|z_p(u_\ell^R[0]) - z_p(u_\ell^L[0])| \leq \frac{G_0}{M^2} |\gamma_\ell(0)| \quad . \tag{6.37B}$$

But by Property (4.2)

$$\sum_{M} |\gamma_\ell(0)| = V_0 \leq V \quad . \tag{6.38}$$

Thus statement (6.36) follows directly from either (6.37A) or (6.37B) by estimating the right hand sides using either (6.30) or (6.34A), respectively.

By (6.26) there exists p, $1 \leq p \leq n$, such that

$$\frac{1}{M} = \sup_x |z_p(u_0^h(x))| \quad .$$

For convenience we assume that

$$\frac{1}{M} = \sup_x \{z_p(u_0^h(x))\} \quad , \tag{6.39}$$

In the case that z is a coordinate system of Riemann invariants, assume that V is arbitrary and that

(6.31) $$V_0 < V \quad ,$$

and

(6.32) $$M > \tilde{M} \quad ,$$

where $\tilde{M}$ is sufficient for U and V in Lemma (GL3). Under these assumptions, if

(6.33) $$h < \mathrm{Min}\left\{ \frac{T\delta^2}{G_0 M^2} , \left(\frac{T^{1/2}\lambda}{30\sqrt{2}\ G_0^{1/2}} \right) , \left(\frac{T^{1/2}}{(12\sqrt{2}\ M)G_0^{3/2}} \right)^2 \right\} \quad ,$$

then

(6.34) $$Q(0) - Q(T) > \frac{1}{(GM)^2} \quad ,$$

where Q is the quadratic Glimm functional associated with u^h [cf (3.11)]. Note that by (6.25), any u^h that satisfies either (6.29) or (6.31), (6.32) must also satisfy

$$\| u^h(\cdot,t) \|_S < \frac{1}{M_2}$$

for $i = 1,2,$ so in particular

(6.34A) $$M > \frac{G_0^2}{\delta} \quad .$$

The remainder of this section is devoted to the proof of Theorem (6.3). From here on out assume that u^h is a given random choice method approximate solution that satisfies (6.1), (6.33) and either (6.29), (6.30) or (6.30), (6.31). Let Γ and $\mathcal{M}$ denote the characteristics and index set associated with u^h and the time level t_J, where

(6.35) $$J \equiv \mathrm{Min}\left\{ \left[\frac{T}{k}\right] + 1,\ \left[\frac{2T}{k}\right] \right\} \quad .$$

(Here $[\ ]$ denotes "greatest integer in".)

The choice of z_p for the p-wave strength parameter determines the definition of wave strength for the characteristics in Γ [cf (4.6)]. In this case, for $\ell \in \mathcal{M}_p$,

$$\gamma_\ell(t) \equiv z_p(u_\ell^R[t]) - z_p(u_\ell^L[t])$$

defines the signed strength of the characteristic γ_ℓ at time $t < t_J$. Recall that $\gamma_\ell(t) \equiv \gamma_\ell$ is constant for $t \in [t_\ell^0, t_\ell^1)$, and identically zero elsewhere.

The idea in the proof to follows is this: since $\| u_0(\cdot) \|_S = \frac{1}{M}$ and $\| u_0(\cdot) \|_0 = \varepsilon$, there must be a "spike" in the initial data of height $\frac{1}{M}$ and width on the order of εM.

In the analysis to follow, it is important to bound changes in λ_p by changes in z_p. This can be done because $z \to u$ is a regular map, and $\nabla\lambda_p \cdot R_p > 0$. Define

(6.20)
$$\lambda \equiv \operatorname*{Inf}_{\sigma, u^L} \frac{\partial\lambda_p(T_p(\sigma; u^L))}{\partial\sigma}$$

where $\sigma = z_p(u) - z_p(u^L)$ and the infemum is taken over all values of σ, u^L such that $u = T_p(\sigma; u^L) \in U$.

Now let G_0 be large enough to satisfy all previous conditions, as well as

(6.21)
$$G_0 > \operatorname{Max}\{1, \lambda^{-1}, V\} \quad .$$

Define the following constants:

(6.22)
$$\delta \equiv \delta(G_0) = \frac{1}{64G_0^2} ,$$

(6.23)
$$G \equiv G(\delta, G_0) = \frac{1}{\delta^2} ,$$

(6.24)
$$M_2 \equiv M_2(\delta, G_0) = \frac{G_0^2}{\delta} \quad .$$

Choose U to be a sufficiently small neighborhood of $u = 0$ so that (6.14) holds and

(6.25)
$$U \subseteq \{u : |u| < M_i^{-1}\}$$

for $i = 1,2$. We prove the follow theorem which is a restatement of Lemma (2) of the Introduction.

<u>THEOREM (6.3)</u>: Let u^h be a fixed random choice method approximate solution satisfying (1). Define

(6.26)
$$\frac{1}{M} \equiv \|u_0^h(0)\|_S ,$$

(6.27)
$$\varepsilon \equiv \|u_0^h(0)\|_{L^1} ,$$

(6.28)
$$T \equiv \varepsilon(GM)^2 \quad .$$

In the case that z is not a coordinate system of Riemann invariants, assume that

(6.29)
$$V_0 < V ,$$

where V is sufficient for U in Lemma (GL2) and

(6.30)
$$V < \frac{\delta}{G_0^2} \quad .$$

$$(6.13) \qquad |z_{p'}(u^R) - z_{p'}(u^L)| < \frac{G_0}{M^2}|\gamma^p| \quad .$$

Proof: Statements (6.12) and (6.13) follow directly from (6.3) together with the fact that $S_p(u^L)$ makes C^2 P.W. C^3 contact with $R_p(u^L)$ at the state u^L [cf. (2.1)].

It is clear from (13) that if U is sufficiently small, then

$$(6.14) \qquad |z_p(u^R) - z_p(u^L)| > 2|z_{p'}(u^R) - z_{p'}(u^L)|$$

for all p-waves with left and right states u^L and u^R in U, $p' \neq p$.

The next lemma is a technical but elementary uniform estimate for the speed of a p-shock in terms of z_p.

LEMMA (6.2): Let S denote a p-shock with speed s and left and right states u^L and u^R. Then there exists a constant $M_1 > 0$ and a constant $G_0 > 0$ depending only on M_1 and f, such that, if $M > M_1$ and

$$(6.15) \qquad |u^q| < \frac{1}{M} \quad , \quad q = L,R \quad ,$$

$$(6.16) \qquad z_p(\bar{u}) - \frac{z_p(u^L) + z_p(u^R)}{2} > \frac{1}{16M} \quad ,$$

and

$$(6.17) \qquad |z_{p'}(\bar{u}) - z_{p'}(u^R)| < \frac{1}{G_0 M} \quad ,$$

for all $p' \neq p$, then

$$(6.18) \qquad \lambda_p(\bar{u}) - s > \frac{1}{G_0 M} \quad .$$

Proof: Lemma (6.2) expresses in a uniform way the fact of Lemma (2.1) that

$$(6.19) \qquad s = \lambda_p\left(\frac{u^L + u^R}{2}\right) + O(1)\;|S|^2 \quad .$$

The existence of M_1 follows from (6.3) and the uniformity follows from the compactness of U together with the fact that

$$\frac{\partial}{\partial z_p}\lambda_p > 0 \quad .$$

satisfying (6.1) together with

$$V_0 \equiv \sum_{i,p} |\gamma_{i0}^p| < V \quad , \tag{6.5}$$

takes values in U for all x and t, and satisfies the conclusions (3.18) - (3.20) of Lemma (GL2) as well. We say that $\tilde{M}$ is sufficient for U and V in Lemma (GL3) if

$$V_0 < V \quad , \tag{6.6}$$

together with

$$\|u_0^h(\cdot)\|_S < \frac{1}{\tilde{M}} \quad , \tag{6.7}$$

guarantee that u^h takes values in U for all x, t, and satisfies the conclusions (3.18) - (3.23) of Lemma (GL3) as well. It is clear from the statements of Lemmas (GL2) and (GL3) that for every neighborhood U of $u = 0$ there is a V' such that any $V < V'$ will be sufficient for U in Lemma (GL2); and in the case of a coordinate system of Riemann invariants, for every U and V there is an $\tilde{M}'$ such that any $\tilde{M} > \tilde{M}'$ will be sufficient for U and V in Lemma (GL3).

For $u \in U$, let $|u| \equiv \sup_p |z_p(u)|$, and for functions $u : R \to U$, define

$$|u(x)| \equiv \sup_{p,x} |z_p(u(x))| \quad , \tag{6.8}$$

$$\|u(\cdot)\|_{L^1} \equiv \int_{-\infty}^{\infty} |u(x)|dx \quad , \tag{6.9}$$

$$\|u(\cdot)\|_S \equiv \sup_x \{|u(x)|\} \quad . \tag{6.10}$$

The following two lemmas will be needed. The first lemma estimates the change in $z_{p'}$ across a p-wave, $p' \neq p$:

<u>LEMMA</u> (6.1): Let γ^p be any p-wave with left and right states u^L and u^R satisfying

$$|u^q| < \frac{1}{M} \quad , \quad q = L, R \quad . \tag{6.11}$$

Then

$$|z_{p'}(u^R) - z_{p'}(u^L)| < \frac{G_0}{M} |\gamma^p| \quad . \tag{6.12}$$

Moreover, if z is a co rdinate system of Riemann invariants, then

§6. THE MAIN ESTIMATE

In this section we study approximate solutions $u^h(x,t)$ generated by the random choice method from initial data $u_0^h(x)$ which satisfies

$$u_0^h(\pm\infty) = 0 \quad . \tag{6.1}$$

We study the approximate solutions in a coordinate system of Riemann invariants if one exists, and if not, then in a coordinate system that is a good approximation to a coordinate system of Riemann invariants near $u = 0$.

Thus, let $z \equiv (z_1,\dots,z_n)$ denote a coordinate system of Riemann invariants if one exists; i.e., in this case assume that the mapping $u \to z$ is a $1-1$ smooth map taking $0 \to 0$, and which satisfies the condition

$$\frac{\partial}{\partial z_k} \propto R_k \quad . \tag{6.2}$$

Such a coordinate system exists if and only if there is a choice of eigenvector fields $\{R_k'\}_{k=1}^n$ such that

$$[R_j', R_k'] = 0$$

for all $j, k \in \{1,\dots,n\}$, where $[\ ]$ denotes the Lie Bracket. A coordinate system of Riemann invariants always exists in the case $n = 2$.

If a coordinate system of Riemann invariants does not exist, then choose $z = (z_1,\dots,z_n)$ to satisfy [cf. 6, 17]

$$\frac{\partial}{\partial z_k}\bigg| = R_k(0) \quad . \tag{6.3}$$

In either case, let z_p be the wave strength parameterization of the p-shock-rarefaction curve $\gamma_p(u_L)$ in a neighborhood of $u = 0$ [cf. (2.1)].

For example, we take $\sigma \equiv z_p(u) - z_p(u_L)$ in the equation $u = T_p(\sigma; u_L)$, so that

$$\gamma^p \equiv z_p(u^R) - z_p(u^L) \tag{6.4}$$

defines the signed strength of a p-wave γ^p with left and right states u^L and u^R.

We let U denote a neighborhood of $u = 0$ in which Riemann problems are uniquely solvable such that z_p is a regular wave strength parameter for all p-wave curves in U, and such that Lemma (GL1) [cf. (3.15), (3.16)] applies with this measure of wave strength. We say that V is sufficient for U in Lemma (GL2) if any approximate solution u^h

thus verifying (5.60). For the best equidistributed sequences, $N(M,\underset{\sim}{a}) = G_0\mu^{-1}$ by Lemma (5.11), and so in this case

$$\delta(\mu) = G_0\mu^2$$

by (5.63) and (5.64). This completes the proof of Theorem (5.12).

$$|E_\ell(1,N)| = |\sum_{j=1}^{N} E_\ell(j)|$$

$$(5.71) \qquad \leq (\frac{k}{h}\lambda_1)h\bar{N}_1 - (1-\frac{k}{h}\lambda_1)h\bar{N}_2 + \frac{k}{h}A_0hN$$

$$\leq (\frac{k}{h}\lambda_1)hN_1 - (1-\frac{k}{h}\lambda_1)hN_2 + hN_3 + \frac{k}{h}A_0hN \quad .$$

But since $N = N(\mu,\ell)$,

$$N_1 \leq (1-\frac{k}{h}\lambda_1)N + \frac{1}{M}N$$

$$N_2 \geq (\frac{k}{h}\lambda_1)N - \frac{1}{M}N$$

$$N_3 \leq 2A_0\frac{k}{h}N + \frac{1}{M}N \quad .$$

Substituting into (5.71) gives

$$|E_\ell(1,N)| \leq (\frac{k}{h}\lambda_1)h(1-\frac{k}{h}\lambda_1)N + (\frac{k}{h}\lambda_1)\frac{1}{M}hN$$

$$- (1-\frac{k}{h}\lambda_1)h(\frac{k}{h}\lambda_1)N + (1-\frac{k}{h}\lambda_1)\frac{1}{M}hN$$

$$+ 2\frac{k}{h}A_0hN + \frac{1}{M}hN + \frac{k}{h}A_0hN$$

$$\leq 2\frac{1}{M}hN + 3\frac{k}{h}A_0hN \quad .$$

Now since $hN = \frac{t_J}{M}$, conclude that

$$|E_\ell(1,N)| \leq t_J\frac{1}{M}\{2\frac{1}{M} + 3\frac{k}{h}A_\ell(0,N)\} \quad .$$

Similarly,

$$|E_\ell(mN+1,(m+1)N)| \leq t_J\frac{1}{M}\{2\frac{1}{M} + 3\frac{k}{h}A_\ell(mN+1,(m+1)N)\} \quad .$$

Therefore,

$$|E_\ell(j_\ell^0,j_\ell^1)| \leq \sum_{m=1}^{M}\{2\frac{1}{M} + 3\frac{k}{h}A_\ell(mN,(m+1)N)\}t_J\frac{1}{M}$$

$$= \{2 + 3\frac{k}{h}A_\ell(j_\ell^0,j_\ell^1)\}t_J\frac{1}{M}$$

$$\leq \{2 + 3\frac{k}{h}A\}t_J\frac{1}{M}$$

$$\leq \mu t_J \quad ,$$

$$E_\ell(j1,j2) \equiv x_\ell(t_{j2}) - x_\ell(t_{j1}) - \int_{t_{j1}}^{t_{j2}} \lambda_\ell[t]dt \quad ,$$

and let

$$\lambda_j \equiv \lambda_\ell[t_j-] \quad .$$

By the definition of approximate characteristics,

$$E_\ell(j1,j2) = \sum_{j=j1}^{j2} E_\ell(j) \quad ,$$

where

$$E_\ell(j) = \begin{cases} (\frac{k}{h}\lambda_j)h, & \text{if } a_j < \frac{k}{h}\lambda_j \\ (\frac{k}{h}\lambda_j - 1)h, & \text{if } a_j > \frac{k}{h}\lambda_j \end{cases} .$$

For $j \in [1,N]$,

$$|\lambda_j - \lambda_1| < A_\ell(1,N) \equiv A_0 \quad .$$

Define

$$\overline{N}_1 = \{j < N : a_j < \frac{k}{h}\lambda_j\} \quad ,$$

$$\overline{N}_2 = \{j < N : a_j > \frac{k}{h}\lambda_j\} \quad ,$$

$$I_1 = [0,\ 1 - \frac{k}{h}\lambda_1 - A_0] \quad ,$$

$$I_2 = [1 - \frac{k}{h}\lambda_1 + A_0, 1] \quad ,$$

$$I_3 = [1 - \frac{k}{h}\lambda_1 - A_0,\ 1 - \frac{k}{h}\lambda_1 + A_0] \quad ,$$

and set

(5.70) $$N_n \equiv N(I_n,1,N) \quad , \quad n = 1,2,3 \quad .$$

Then by (5.70)

$$\overline{N}_1 < N_1 + N_3 \quad ,$$

$$\overline{N}_2 < N_2 + N_3 \quad .$$

Now we can estimate

$$h = \frac{\nu}{MN} \quad ,$$

and by definition of δ,

$$N > N(M,\underset{\sim}{a}) \quad .$$

By (5.51),

(5.66) $$\sum_{j'=0}^{j} |\lambda_\ell[t_j,+] - \lambda_\ell[t_j,-]| < A_\ell(1,j) \quad .$$

Thus for some interger m, $0 < m < M-1$, we must have

(5.67) $$A_\ell(mN,(m+1)N) < \frac{A_\ell(0,j_\ell^1)}{M} < \frac{A}{M} \quad .$$

But this implies that for some $j \in [mN,(m+1)N]$,

$$x_i + a_j h \in (x_i + \lambda_L(j)k,\ x_i + \lambda_L(j)k + \frac{3A}{M}) \quad ,$$

or equivalently,

$$a_j \in [\lambda_\ell[t_j],\ \lambda_\ell[t_j] + \frac{3A}{M}]\ \frac{k}{h} \quad ,$$

where $i = \ell(j)$. This follows because (5.66) and (5.67) imply that there is a fixed open interval I, $I \subset [\lambda_L(j),\lambda_L(j) + \frac{3A}{M}]\ \frac{k}{h}$ for all $j \in [mN,(m+1)N]$, such that $|I| > \frac{A}{M} > \frac{1}{M}$, where we use (5.62). Thus by Lemma (5.11),

$$N(I,mN,(m+1)N) > \{|I| - \frac{A}{M}\}\ N > 0 \quad ,$$

and so

$$N(I,mN,(m+1)N) > 1 \quad .$$

By definition of approximate characteristics, this implies that

$$|\lambda_p(u_\ell^R[t_j+]) - \lambda_p(u_\ell^L[t_j-])| < \frac{3A}{M}$$

for that value of $j \in [mN,(m+1)N]$ for which $a_j \in I$. Thus by (5.65),

$$|\gamma_\ell(t_j+)| < 3G_0\ \frac{A}{M} < \nu \quad .$$

This verifies (5.59) since $t_j < \nu < t_\ell^1$ implies $|\gamma_\ell| = |\gamma_\ell(t_j+)|$.

We now verify (5.60). Fix $h < \delta(\mu)t_J$, and set $N \equiv t_J M^{-1}h^{-1}$ so that

$$h = \frac{t_J}{MN} \quad ,$$

and by (5.64)

$$N > N(M,\underset{\sim}{a}) \quad .$$

Without loss of generality, we do the case $\ell \in M(t_J)\setminus N(t_J)$, the case $\ell \in N(t_J)$ being similar. Thus let $\ell \in M(J)\setminus N(t_J)$ be fixed. Define

If

$$h < \delta(\mu)t_J,\ t \in [t_\ell^0, t_\ell^1) \quad , \tag{5.60}$$

then

$$|E_\ell(t)| < \mu t_J \quad ,$$

for all $\ell \in M$.

(Here (5.59) says that the strength of rarefaction characteristics tends to zero uniformly with h due to the splitting of characteristics; and (5.60) says that as $h \to 0$, characteristics move wih characteristic speed.)

Proof: First, for $\ell \in M$, define [cf (5.51), Lemma (5.10)]

$$A_\ell(j1,j2) = \sum_{j=j1}^{j2} G_0\{B_\ell(j) + B_\ell^-(j) + C_{\ell(j),j} + D_{\ell(j),j}\} \quad . \tag{5.61}$$

Set $A \equiv \mathrm{Max}\{2, G_0V_0\}$, so that

$$A_\ell(j1,j2) < G_0V_0 < A,\ A > 2 \quad , \tag{5.62}$$

for all $j1,\ j2 < J$. Let $\underset{\sim}{a}$ and $\mu > 0$ be given. Let M be the smallest integer such that

$$M > 3G_0A(2 + \frac{k}{h})\mu^{-1} \quad , \tag{5.63}$$

and define

$$\delta(\mu) = \frac{1}{MN(M,\underset{\sim}{a})} \quad , \tag{5.64}$$

where in addition to all previous estimates, G_0 satisfies

$$|\gamma_\ell| < G_0|\lambda_p(u_\ell^L[t] - \lambda_p(u_\ell^R[t])| \quad , \tag{5.65}$$

for all $\ell \in M_p^+$, $p = 1,\dots,n$, $t_\ell^0 < t < t_\ell^1 < t_J$. (Recall that the strength $|\gamma_\ell(t)|$ is constant and equal to $|\gamma_\ell|$ for all $t_\ell^0 < t < t_\ell^1$.) Such a G_0 exists by Property (4.3) together with the assumption of genuine nonlinearity. Here $N(M,\underset{\sim}{a})$ is defined in Lemma (5.11). We first verify (5.59). For this case choose $\nu > 0$, and let $\ell \in M_p^+$ satisfy $t_\ell^1 > \nu$. Fix $h < \delta(\mu)\nu$, and set

$$N = \nu M^{-1}h^{-1} \quad ,$$

so that

Theorem (6.3) is a direct consequence of Proposition (6.6) together with

<u>PROPOSITION (6.7)</u>: If $Q(0) - Q(T) < \frac{1}{(GM)^2}$ where u^h satisfies our previous assumptions, then either $r \in N(T)$ or $s \in N(T)$.

<u>Proof of Theorem (6.3)</u>: If $Q(0) - Q(T) < \frac{1}{(GM)^2}$, then by Proposition (6.7) either $r \in N(T)$ or $s \in N(T)$. But by Proposition (6.6) both r and s are in $M \setminus N(T)$. Thus by contradiction we must have $Q(0) - Q(T) \geq \frac{1}{(GM)^2}$.

It remains to give a proof of Proposition (6.7). Proposition (6.7) is a consequence of the following lemmas. The idea is to show that if $Q(0) - Q(T) < \frac{1}{(GM)^2}$, then for $t < T$, $\lambda_r[t]$ and $\lambda_s[t]$ are sufficiently close to $\lambda_r[0]$ and $\lambda_s[0]$, respectively, to guarantee that the characteristics γ_r and γ_s must intersect before time T. Then by Proposition (5.2), $r \in N(T)$ or $s \in N(T)$.

<u>LEMMA (6.8)</u>. Assume that $Q(0) - Q(T) < \frac{1}{(GM)^2}$. Then

$$\sum_{R_0^L} |\gamma_\ell| \geq \frac{\delta}{M} , \tag{6.70}$$

$$\sum_{S_0^R} |\gamma_\ell| \geq \frac{\delta}{M} . \tag{6.71}$$

<u>Proof</u>: By Property (4.2),

$$TV\{z_p(u_0^h(0))\} = \sum_{M} |z_p(u_\ell^R[0]) - z_p(u_\ell^L[0])| .$$

This together with (6.40) - (6.43) and (6.46) implies that

$$\sum_{R^L} |\gamma_\ell| \geq \frac{3}{4M} - \frac{\delta}{M} .$$

(Here we use the fact that $z_p(u_\ell^R[t]) - z_p(u_\ell^L[t])$ is positive [negative] for $\ell \in M_p^+ [\ell \in M_p^-]$, respectively.) But

$$\sum_{R^L} |\gamma_\ell| \geq \sum_{R_0^L} |\gamma_\ell| - \sum_{N(T)} |\gamma_\ell| ,$$

and by Corollary (5.8),

$$\sum_{N(T)} |\gamma_\ell| < \frac{G_0}{GM} < \frac{\delta}{M}$$

since we assume $Q(0) - Q(T) < \frac{1}{(GM)^2}$. Thus by (6.22), $\delta < \frac{1}{32}$, so

$$\sum_{R_0^L} |\gamma_\ell| > \frac{3}{4M} - \frac{2\delta}{M} > \frac{\delta}{M} .$$

Similarly, in the case (6.49), (6.50) of Proposition (6.6), we must have

$$\sum_{S^R} |\gamma_\ell| = \sum_{S^R} |\gamma_\ell| - \sum_{N(T)} |\gamma_\ell|$$

$$> \frac{1}{8M} - \frac{\delta}{M} - \frac{\delta}{M} > \frac{\delta}{M} .$$

Since $s \in S_0^R$, the case (6.47), (6.48) immediately gives the conclusion (6.71). This completes the proof of Lemma (6.8).

<u>LEMMA (6.9A)</u>: If $\ell \in M_{p'} \setminus N(T)$, $p' \neq p$, and $x_\ell(0) \in (x_r(0), x_s(0))$, then in time $[0,T]$ γ_ℓ either intersects all the characteristics in R_0^L (the case $p' < p$) or else it intersects all the characteristics in S_0^R (the case $p' > p$).

<u>LEMMA (6.9B)</u>: If $\ell \in M_{p'} \setminus N(T)$, $p' \neq p$ *and* γ_ℓ *intersects* γ_r or γ_s in time $[0, \frac{T}{2}]$, then in time $[0,T]$, γ_ℓ either intersects all the characteristics in R_0^L or else it intersects all the characteristics in S_0^R.

<u>Proof</u>: Since $\underset{\sim}{a}$ is assumed to be best equidistributed, statement (5.60) of Theorem (5.12) implies that for any $\ell \in M_p \setminus N(T)$, $1 \le p \le n$, and $t \le T$,

$$|x_\ell(t) - x_\ell(0) - \int_0^t \lambda_\ell[t]dt| \le \mu t_J = (G_0 h t_J)^{1/2} .$$

Thus

$$|x_\ell(t) - x_\ell(0) - \lambda_p t| \le (G_0 h t_J)^{1/2} + \frac{G_0 T}{M} \equiv E , \tag{6.72}$$

where without loss of generality we have taken G_0 large enough so that

$$|\lambda_\ell(t) - \lambda_p| \le \frac{G_0}{M}$$

for all $\ell \in M_p(J)$, $t \le t_J$.

For the proof of Lemma (6.9A), assume that $\ell \in M_{p'} \setminus N(T)$, $p' \neq p$. We do the case $p' < p$ and γ_ℓ intersects γ_r in time $[0, T/2]$; i.e., we show that $x_\ell(T) - x_{r'}(T) < 0$ for all $r' \in R_0^L$ [cf. Prop. (5.1)]. By (6.72),

$$
\begin{aligned}
x_\ell(T) - x_{r'}(T) &= x_\ell(T) - x_r(T) + x_r(T) - x_{r'}(T) \\
&< [x_\ell(0) - x_r(0)] + [\lambda_{p'} - \lambda_p]T + 2E \\
&\quad + [x_r(0) - x_{r'}(0)] + [\lambda_p - \lambda_p]T + 2E \quad .
\end{aligned} \tag{6.73}
$$

Since γ_ℓ intersects γ_r in time $[0, T/2]$, (6.72) also implies

$$0 > x_\ell(\tfrac{T}{2}) - x_r(\tfrac{T}{2}) > [x_\ell(0) - x_r(0)] + [\lambda_{p'}(0) - \lambda_p(0)]\,\frac{T}{2} - E \quad .$$

Moreover, by Lemma (6.5),

$$[x_r(0) - x_{r'}(0)] < |x_B - x_A| < \frac{\varepsilon M}{\delta} \quad .$$

Therefore substituting into (6.73) yields

$$
\begin{aligned}
x_\ell(T) - x_{r'}(T) &< [\lambda_{p'}(0) - \lambda_p(0)]\,\frac{T}{2} + \{\frac{\varepsilon M}{\delta} + 5E\} \\
&< -\lambda\,\frac{T}{2} + \{\frac{\varepsilon M}{\delta} + 5(G_0 h t_J)^{1/2} + \frac{5G_0 T}{M}\} \\
&= \{-1 + \frac{2\varepsilon M}{\delta T\lambda} + \frac{10}{T\lambda}\,(G_0 h t_J)^{1/2} + \frac{10G_0}{M\lambda}\}\,\frac{T\lambda}{2} \quad .
\end{aligned} \tag{6.74}
$$

By (6.28) and (6.34A) we have

$$\frac{2\varepsilon M}{\delta T\lambda} < \frac{2}{\delta G^2 M\lambda} < \frac{2}{\delta G^2(\frac{G_0^2}{\delta})\lambda} < \frac{1}{3} \quad ; \tag{6.75}$$

by (6.33), $h < \left(\dfrac{T^{1/2}\lambda}{30\sqrt{2}\;G_0^{1/2}}\right)^2$, so

$$\frac{10}{T\lambda}\,(G_0 h t_J)^{1/2} < \left(\frac{10\sqrt{2}\;G_0^{1/2}}{T^{1/2}\lambda}\right) h^{1/2} < \frac{1}{3} \quad ; \tag{6.76}$$

and by (6.34A) again,

$$\frac{10G_0}{M\lambda} < \frac{1}{3} \quad . \tag{6.77}$$

Thus (6.74) implies

$$x_\ell(T) - x_{r'}(T) < 0 \quad .$$

This completes the proof of Lemma (6.9A).

For the proof of Lemma (6.9B), assume $\ell \in M_{p'} \setminus N(T)$, $p \neq p'$. We do the case $p' < p$; i.e., we show that if $\ell \in M_{p'}$, $p' < p$, and $x_\ell(0) \in (x_r(0), \bar{x})$, then γ_ℓ intersects $\gamma_{r'}$ in time $[0,T]$ for all $r' \in R_0^L$. It suffices to show that $x_\ell(T) - x_{r'}(T) < 0$.

$$x_\ell(T) - x_{r'}(T) \le [x_\ell(0) - x_{r'}(0)] + [\lambda_p' - \lambda_p]T + 2E$$

$$\le \frac{\varepsilon M}{\delta} - \lambda T + 2E$$

$$= \left\{-1 + \frac{\varepsilon M}{\delta T\lambda} + \frac{2}{T\lambda}(G_0 h T_J)^{1/2} + \frac{2G_0}{M\lambda}\right\} T\lambda < 0$$

where we have applied (6.75) - (6.77). This completes the proof of Lemma (6.9B).

<u>LEMMA (6.10)</u>. Assume that $Q(0) - Q(T) < \frac{1}{(GM)^2}$. Then for $q = 0$, r and s (cf. (6.63) - (6.69))

(6.78) $$B_q < \frac{\delta}{M} \quad .$$

<u>Proof</u>. Write

$$B_q \equiv \sum_{B_q} |\gamma_\ell| = \sum_{B_q \setminus N(T)} |\gamma_\ell| + \sum_{B_q \setminus N(T)} |\gamma_\ell| \quad .$$

By Corollary (5.8),

(6.79) $$\sum_{B_q \cap N(T)} |\gamma_\ell| < \frac{G_0}{GM} \quad ,$$

since we assume that $Q(0) - Q(T) < \frac{1}{(GM)^2}$. On the other hand, if $\ell \in B_q \setminus N(T)$, then Lemma (6.9) implies that γ_ℓ intersects all the characteristics in either R_0^L or S_0^R in time $[0,T]$. For example, assume the case R_0^L, and define

$$A' \equiv \{\langle \ell, r' \rangle : \ell \in B_q \setminus N(T) \text{ and } r' \in R_0^L\} \quad .$$

Then $A' \subseteq A(0) \setminus A(T)$. Thus Lemma (5.7) implies

(6.80) $$\sum_{A'} |\gamma_\ell|\, |\gamma_{r'}| < \frac{G_0}{G^2 M^2} \quad ,$$

since we assume $Q(0) - Q(T) < \frac{1}{(GM)^2}$. But by Lemma (6.8),

$$\sum_{A'} |\gamma_\ell| \ |\gamma_{r'}| = \frac{1}{2} \{ \sum_{\mathcal{B}_q \setminus N(T)} |\gamma_\ell| \} \{ \sum_{R_0^L} |\gamma_\ell| \}$$

(6.81)

$$\geq \frac{\delta}{2M} \sum_{\mathcal{B}_q \setminus N(T)} |\gamma_\ell| \quad .$$

Combining (6.80) and (6.81) gives

(6.82)
$$\sum_{\mathcal{B}_q \setminus N(T)} |\gamma_\ell| < \frac{2G_0}{\delta G^2 M} \quad ;$$

and combining (6.80) and (6.82) gives

$$B_q < \frac{4G_0}{\delta G^2 M} < \frac{\delta}{M}$$

where we have applied (6.22) and (6.23). This completes the proof of Lemma (6.10).

<u>LEMMA (6.11)</u>: If $Q(0) - Q(T) < \frac{1}{(GM)^2}$, then

(6.83)
$$B_{sq}^- < \frac{\delta}{M} \quad ,$$

for $q = L$ and R.

<u>Proof</u>. First assume that (6.48) of Proposition (6.6) holds, and write

$$B_s^- = \sum_{\mathcal{B}_s^-} |\gamma_\ell| = \sum_{\mathcal{B}_s^- \cap N(T)} + \sum_{\mathcal{B}_s^- \setminus N(T)} \quad .$$

By Corollary (5.8),

$$\sum_{\mathcal{B}_s^- \cap N(T)} |\gamma_\ell| < \frac{G_0}{GM} \quad ,$$

and moreover (as in the proof of Lemma (6.10)), $A' \subseteq A(0) \setminus A(T)$ where

$$A' \equiv \{ \langle \ell, s' \rangle : \ell \in \mathcal{B}_s^- \ , \ s' \in S_0^R, \ x_{s'}(0) = x_s(0) \} \quad .$$

(Here we apply Lemma (5.2).) Thus as in (6.80) - (6.82), Lemma (5.7) implies that

$$\mathcal{B}_{sq}^- < \mathcal{B}_s^- < \frac{4G_0}{\delta G^2 M} < \frac{\delta}{M} \ , \ q = L, R \quad ,$$

where we apply (6.48) in place of Lemma (6.8).

Now assume (6.49) - (6.52) of Proposition (6.6) holds. In this case (6.50) - (6.52) satisfy the hypotheses (5.42), (5.43) of Proposition (5.9) with $L = \frac{\delta}{M}$. Since we assume $Q(0) - Q(T) < \frac{1}{(GM)^2}$, and $\frac{1}{(GM)^2} = \delta^2(\frac{\delta}{M})^2 < G_0(\frac{\delta}{M})^2$, we can conclude

$$\beta^-_{sq} < \frac{\delta}{M}$$

for $q = L,R$. This completes the proof of Lemma (6.11).

<u>LEMMA (6.12):</u> Assume that $Q(0) - Q(T) < \frac{1}{(GM)^2}$. Then

$$\lambda_r[0] - \lambda_s[0] > \frac{1}{G_0 M} \quad . \tag{6.84}$$

<u>Proof</u>. We prove Lemma (6.12) by satisfying the hypotheses of Lemma (6.2). First, (6.46) implies

$$z_p(u^L_r[0]) > \frac{3}{4M} \quad , \tag{6.85}$$

and (6.47), (6.49) imply

$$z_p(u^R_s[0]) < \frac{3}{8M} \quad . \tag{6.86}$$

Let $\bar{u} \equiv u^L_r[0]$, and let S denote the p-shock

$$S \equiv \gamma^p_{i_s 0} \quad .$$

Let $\{u^L, u^R\}$ denote the left and right states, and σ the speed of the p-shock S. By (5.49) $\lambda_s[0] = \sigma$, and (6.86) gives

$$0 < z_p(u^R) < z_p(u^R_s[0]) < \frac{3}{8M} \quad .$$

By (6.39) we must have

$$z_p(u^L) < \frac{1}{M} \quad .$$

Thus

$$\frac{z_p(u^L) + z_p(u^R)}{2} < \frac{11}{8M} \quad .$$

By (6.85)

$$z_p(\bar{u}) - \frac{z_p(u^L) + z_p(u^R)}{2} > \frac{1}{16M} \quad .$$

Moreover, for $p' \neq p$, the difference in $z_{p'}$ between u^R and $\bar{u}$ is bounded by the total variation in $z_{p'}$ of all waves that lie between $x_r(0)$ and $x_s(0)$ at $t = 0$. By Property (4.3), this can be estimated by

$$|z_{p'}(\bar{u}) - z_{p'}(u^R)| < B_0 + \sum_{M_p} |z_{p'}(u_\ell^R[0]) - z_{p'}^L(u_\ell^L[0])|$$

$$< \frac{\delta}{M} + \frac{\delta}{M} = \frac{2\delta}{M} \quad ,$$

where we have applied Lemma (6.4) and Lemma (6.10) in the second inequality. Thus Lemma (6.2) applies with $L = \frac{1}{M}$ to give

$$\lambda_p(\bar{u}) - \sigma > \frac{1}{G_0 M} \quad .$$

Therefore we conclude

$$\lambda_r[0] - \lambda_s[0] = \lambda_p(\bar{u}) - \sigma > \frac{1}{G_0 M} \quad .$$

Proof of Proposition (6.7): Assume that $Q(T) - Q(0) < \frac{1}{(GM)^2}$. We show that γ_r intersects γ_s in time $[0, T/2]$. By Proposition (5.2) this implies that either $r \in N(T)$ or $s \in N(T)$. Thus it suffices to show that $x_s(T/2) - x_r(T/2) < 0$. By (5.53),

$$|\lambda_s[T/2] - \lambda_s[0]| < G_0\{B_s + B_s^- + [Q(0)-Q(T)]^{1/2}\}$$

(6.87)
$$< G_0\{\frac{\delta}{M} + \frac{\delta}{M} + \frac{1}{GM}\} < \frac{3G_0\delta}{M} \quad ,$$

where we applied Lemmas (6.10) and (6.11). Therefore we can use (5.42) and (5.44) to obtain

$$x_s(T/2) < x_s(0) + \int_0^{T/2} \lambda_s(t)dt + (G_0 h t_J)^{1/2}$$

(6.88)
$$< x_s(0) + \lambda_s[0]\,\frac{T}{2} + \frac{3\delta G_0}{M}\,\frac{T}{2} + (G_0 h t_J)^{1/2} \quad .$$

Similarly,

(6.89)
$$x_r(T/2) > x_r(0) + \lambda_r[0]\,\frac{T}{2} - \frac{3\delta G_0}{M}\,\frac{T}{2} - (G_0 h t_J)^{1/2} \quad .$$

Subtracting (6.89) from (6.88) gives

$$x_s(T/2) - x_r(T/2) < [x_s(0) - x_r(0)] + \{\lambda_s[0] - \lambda_r[0]\}\,\frac{T}{2}$$

(6.90)
$$+ 2\{\frac{3\delta G_0 T}{2M} + (G_0 h t_J)^{1/2}\} \quad ;$$

and by Lemmas (6.5) and (6.12) respectively,

$$|x_s(0) - x_r(0)| < \frac{\varepsilon M}{\delta} \quad ,$$

$$\lambda_s[0] - \lambda_r[0] < -\frac{1}{G_0 M} \quad .$$

Therefore

$$x_s(T/2) - x_r(T/2) < \left\{\frac{\varepsilon M}{\delta} + \frac{3\delta G_0 T}{M} + 2(G_0 h t_J)^{1/2} - \frac{T}{2G_0 M}\right\}$$

(6.91)

$$= \left\{\frac{2G_0 M^2 \varepsilon}{\delta T} + 6G_0^2\delta + \frac{4G_0 M (G_0 h t_J)^{1/2}}{T} - 1\right\} \frac{T}{2G_0 M} \quad .$$

By (6.28) and (6.22), (6.23),

(6.92)
$$\frac{2G_0 M^2 \varepsilon}{\delta T} = \frac{2G_0 M^2 \varepsilon}{\delta \varepsilon G^2 M^2} = \frac{2G_0}{\delta G^2} = 2\delta^3 G_0 < \frac{1}{3} \quad .$$

By (6.22)

(6.93)
$$6G_0^2\delta < \frac{1}{3} \quad .$$

By (6.28) and (6.33),

(6.94)
$$\frac{4G_0 M (G_0 h t_J)^{1/2}}{T} < \frac{4\sqrt{2}\, M G_0^{3/2} h^{1/2}}{T^{1/2}} < \left(\frac{4\sqrt{2}\, M G_0^{3/2}}{T^{1/2}}\right)\left(\frac{T^{1/2}}{12\sqrt{2}\, M G_0^{3/2}}\right) < \frac{1}{3} \quad .$$

Finally, putting (6.92) - (6.94) into (6.91) yields

$$x_s(T/2) - x_r(T/2) < 0 \quad .$$

This completes the proof of Proposition (6.7), and hence also completes the proof of Theorem (6.3). The proof of Theorem (6.3) also applies to the case of periodic data, in which case $\| \;\|_{L^1}$, $\| \;\|_S$ and $Q(t)$ are defined on each period.

REFERENCES

[1] Chern, I-Liang, On the decay of a strong wave for hyperbolic conservation laws in one space dimension, thesis, Courant Institute (1982).

[2] Courant, C. and Friedrichs, K. O., "Supersonic Flow and Shock Waves", Springer-Verlag, 1948.

[3] Dafermos, C., Application of the invariance principle for compact processes, II. Asymptotic behavior of solutions of a hyperbolic conservation law, J. Differential Equations, 11 (1972), 416-424.

[4] DiPerna, R., Decay and asymptotic behavior of solutions to nonlinear hyperbolic systems of conservation laws, Indiana Univ. Math. J., 24 (1975), 1047-1071.

[5] __________, Decay of solutions of hyperbolic systems of conservation laws with a convex extension, Arch. Ratl. Mech.'s Anal., 69 (1977), 1-46.

[6] Glimm, J., Solutions in the large for nonlinear hyperbolic systems of equations, Comm. Pure Appl. Math., 18 (1965), 695-715.

[7] Glimm, J. and Lax, P., Decay of solutions of systems of nonlinear hyperbolic conservation laws, Amer. Math. Soc. Memoirs, 101 (1970).

[8] Keyfitz, B., *Solution with shocks: an example of an L^1-contractive semigroup*, Comm. Pure Appl. Math., 24, (1971), 125-132.

[9] Lax, P., Hyperbolic systems of conservation laws, II, Comm. Pure Appl. Math., 10 (1957), 537-566.

[10] Liu, T. P., Asymptotic behavior of solutions of general system of nonlinear hyperbolic conservation laws, Indiana Univ. J., 27 (1978), 211-253.

[11] _______, Admissible solutions of hyperbolic conservation laws, Memoirs of the AMS, 30, No. 240 (1981).

[12] _______, Decay to N-waves of solutions of general system of nonlinear hyperbolic conservation laws, Comm. Pure Appl. Math., 30 (1977), 585-610.

[13] _______, Large-time behavior of solutions of initial and initial-boundary value problem of general system of hyperbolic conservation laws, Comm. Math. Phys., 55 (1977), 163-177.

[14] _______, Linear and nonlinear large-time behaviors of solutions of hyperbolic conservation laws, Comm. Pure Appl. Math., 30 (1977), 767-796.

[15] _______, The deterministic version of the Glimm scheme, Comm. Math. Phys., 57 (1977), 135-148.

[16] Luskin, L. and Temple, J. B., "The Existence of a Global Weak Solution to the Nonlinear Waterhammer Problem", Comm. Pure Appl. Math., 35 (1982), 697-735.

[17] Smoller, J. A., Shock waves and reaction-diffusion equations, Springer-Verlag (1980).

[18] Temple, J. B., "Global Solution of the Cauchy Problem for a Class of 2×2 Nonstrictly Hyperbolic Conservation Laws", Adv. Appl. Math., 3 (1982), 335-375.

[19] _____________, "Solutions in the Large for the Nonlinear Hyperbolic Conservation Laws of Gas Dynamics", J. Diff. Equs., Vol. 41, No 1 (1981), 96-161.

[20] _____________, "Systems of Conservation Laws With Coinciding Shock and Rarefaction Curves", Contemporary Mathematics, Vol. 17 (1983), 143-151.

[21] _____________, "Systems of Conservation Laws With Invariant Submanifolds", Trans. Am. Math. Soc., Vol. 280, No. 2 (1983), 781-795.

[22] _____________, "No L^1-contractive metrics for systems of Conservation Laws", Trans. Am. Math. Soc., Vol 288, No. 2, April 1985.

[23] _____________, "On Supnorm Bounds in Glimm's Method", Mathematics Research Center Technical Summary Reports, (to appear), University of Wisconsin-Madison.

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STRACT (Continue on reverse side if necessary and identify by block number)

We show that solutions of the Cauchy problem for systems of two conservation decay in the supnorm at a rate that depends only on the L^1 norm of the ial data. This implies that the dissipation due to the entropy dominates the inearities in the problem at a rate depending only on the L^1 norm of the ial data. Our results apply to any BV initial data u_0 satisfying $\infty) = 0$, and $\mathrm{Sup}\{u_0(\cdot)\} << 1$. The problem of decay with a rate independent ie support of the initial data is central to the issue of continuous

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ndence in systems of conservation laws because of the scale invariance of equations. Indeed, our result implies that the constant state is stable respect to perturbations in $L^1_{\ell oc}$. This is the first stability result in L^p norm for systems of conservation laws. It is crucial that we estimate y in the supnorm since the total variation does not decay at a rate independent he support of the initial data.

The main estimate requires an analysis of approximate characteristics for proof. A general framework is developed for the study of approximate acteristics, and the main estimate is obtained for an arbitrary number of tions.

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